# Suggested Solution to Assignment 7

## Exercise 7.1

1. Suppose there exists one non-constant harmonic function u in D, which attains its maximum M at  $\mathbf{x}_0 \in D$ . Then by the mean value property, we have

$$u(\mathbf{x}_0) = \frac{1}{|B(\mathbf{x}_0, r)|} \iint_{B(\mathbf{x}_0, r)} u dS,$$

for all  $\overline{B(\mathbf{x}_0,r)} \subset D$ . Thus  $u(\mathbf{x}) = M$  for all  $\mathbf{x} \in \overline{B(\mathbf{x}_0,r)} \subset D$  since u is continuous and M is its maximum.

Now since u is not a constant, there exists  $\mathbf{x}_1 \in D$  such that  $u(\mathbf{x}_1) \neq u(\mathbf{x}_0)$ . Since D is a region, we can find a continuous a curve  $\gamma(t) \subset D$  ( $0 \leq t \leq 1$ ) such that  $\gamma(0) = \mathbf{x}_0$  and  $\gamma(1) = \mathbf{x}_1$ . Set  $E := \{0 \leq t \leq 1 | u(\gamma(t)) = M\}$ , then E is closed since u and  $\gamma$  are both continuous. Let  $t_0 \in E$ , then by the above result,  $u(\mathbf{x}) = M$  for all  $\mathbf{x} \in \overline{B(\gamma(t_0), r)} \subset D$ . So E is open. Hence, E = [0, 1] since  $E \neq \emptyset$  and [0, 1] is connected. But this contradicts to  $u(\mathbf{x}_1) \neq u(\mathbf{x}_0)$  and then we complete the proof.  $\Box$ 

2. Suppose  $u_1$  and  $u_2$  are solutions of the problem. Then  $u = u_1 - u_2$  satisfies

$$\Delta u = 0$$
 in  $D$ ,  $\frac{\partial u}{\partial n} = 0$  on  $\partial D$ .

Thus, by the Green's first identity for v = u, we have

$$\iint_{\partial D} u \cdot \frac{\partial u}{\partial n} \, dS = \iiint_D |\nabla u|^2 d\mathbf{x} + \iiint_D u \Delta u d\mathbf{x}.$$

Hence,

$$\iiint_D |\nabla u|^2 d\mathbf{x} = 0.$$

 $\Box$ .

Therefore,  $\nabla u \equiv 0$  and then u(x) is a constant.

3. Suppose  $u_1$  and  $u_2$  are solutions of the problem. Then  $u = u_1 - u_2$  satisfies

$$\Delta u = 0$$
 in  $D$ ,  $\frac{\partial u}{\partial n} + au = 0$  on  $\partial D$ .

Thus, by the Green's first identity for v = u, we have

$$\iint_{\partial D} u \cdot \frac{\partial u}{\partial n} \, dS = \iiint_D |\nabla u|^2 d\mathbf{x} + \iiint_D u \Delta u d\mathbf{x}.$$

Hence,

$$\iiint_D |\nabla u|^2 d\mathbf{x} = -\iint_{\partial D} a u^2 dS.$$

Since  $a(\mathbf{x}) > 0$ ,  $\nabla u \equiv 0$  and then u(x) is a constant C. So by the Robin boundary conditions, we have  $Ca(\mathbf{x}) = 0$ . This shows that C = 0 and  $u \equiv 0$ .  $\Box$ .

4. Suppose  $u_1$  and  $u_2$  are solutions of the problem. Then  $u = u_1 - u_2$  satisfies

$$u_t - k\Delta u = 0$$
 in  $D \times (0, \infty)$ ,  $u = 0$  on  $\partial D \times (0, \infty)$ ,  $u(x, 0) = 0$  in  $D$ .

Thus, by the Green's first identity for v = u, we have

$$\iint_{\partial D} u \cdot \frac{\partial u}{\partial n} \, dS = \iiint_D |\nabla u|^2 d\mathbf{x} + \iiint_D u \Delta u d\mathbf{x}.$$

Therefore, by the diffusion equation

$$0 = \iiint_D |\nabla u|^2 d\mathbf{x} + \frac{1}{k} \iiint_D u u_t d\mathbf{x}$$
$$= \iiint_D |\nabla u|^2 d\mathbf{x} + \frac{1}{2k} \frac{d}{dt} \iiint_D u^2 d\mathbf{x}.$$

Define  $E(t) := \iiint_D u^2 d\mathbf{x}$ , then the above equality implies  $\frac{d}{dt}E(t) \le 0$ . Note that E(0) = 0 and  $E(t) \ge 0$ , we have  $E(t) \equiv 0$ . So we obtain  $u \equiv 0$  and the uniqueness for the diffusion equation with Dirichlet boundary conditions is proved.  $\Box$ .

5. Suppose  $u(\mathbf{x})$  minimizes the energy. Let  $v(\mathbf{x})$  be any function and  $\epsilon$  be any constant, then

$$E[u] \le E[u + \epsilon v] = E[u] + \epsilon \iiint_D \nabla u \cdot \nabla v d\mathbf{x} - \epsilon \iint_{\partial D} hv dS + \epsilon^2 \iiint_D |\nabla v|^2 d\mathbf{x}$$

Hence, by calculus

$$\iiint_D \nabla u \cdot \nabla v d\mathbf{x} - \iint_{\partial D} hv dS = 0$$

for any function v. By Green's first identity,

$$-\iiint_D \nabla u \cdot v d\mathbf{x} + \iint_{\partial D} v \cdot \frac{\partial u}{\partial n} dS - \iint_{\partial D} h v dS = 0$$

Noting v is an arbitrary function, let v vanish on  $\partial D$  firstly and then the above equality implies  $-\Delta u = 0$  in D. So the above equality changes to

$$\iint_{\partial D} v \cdot \frac{\partial u}{\partial n} dS - \iint_{\partial D} hv dS = 0.$$

Since v is arbitrary,  $\frac{\partial u}{\partial n} = h$  on  $\partial D$  and then  $u(\mathbf{x})$  is a solution of the following Neumann problem

$$-\Delta u = 0$$
 in  $D$ ,  $\frac{\partial u}{\partial n} = h$  on bdy  $D$ .

Note that the Neumann problem has a unique solution up to constant and the functional E[w] does not change if a constant is added to w, thus it is the only function that can minimize the energy up to a constant.

Note that here we assume functions u, v and the domain D are smooth enough, at least under which the Green's first identity can be hold.  $\Box$ 

6. (a) Suppose  $u_1$  and  $u_2$  are solutions of the problem. Then  $u = u_1 - u_2$  tends to zero at infinity, and is constant on  $\partial A$  and constant on  $\partial B$ , and satisfies

$$\iint_{\partial A} \frac{\partial u}{\partial n} dS = 0 = \iint_{\partial B} \frac{\partial u}{\partial n} dS.$$

Suppose that  $u \neq 0$ , then without loss of generality, we assume that there exists  $\mathbf{x}_0 \in \overline{D}$  such that  $u(\mathbf{x}_0) > 0$ . Since u tends to zero at infinity, so there exists  $R \gg 1$  such that  $u(\mathbf{x}) \leq \frac{u(\mathbf{x}_0)}{2}$  if  $|\mathbf{x}| \geq R$ . Then u is a harmonic function in  $D \cap B(\mathbf{0}, R)$  and  $\max_{\partial B(\mathbf{0}, R)} u \leq \frac{u(\mathbf{x}_0)}{2} \leq u(\mathbf{x}_0) \leq \max_{D \cap B(\mathbf{0}, R)} u$ . Thus, the maximum is attained on  $\partial A$  or  $\partial B$  by the Maximum Principle. WOLG, we assume that u attain its maximum on  $\partial A$  and then by any point of  $\partial A$  is a maximum point since u is constant on  $\partial A$ . Hence, by the Hopf maximum principle,  $\frac{\partial u}{\partial n} > 0$  on  $\partial A$ , but this contradicts to the condition

$$\iint_{\partial A} \frac{\partial u}{\partial n} dS = 0.$$

Therefore,  $u \equiv 0$  and then the solution is unique.

- (b) If not, then  $u(\mathbf{x})$  have a negative minimum. As above, by the Minimum Principle, we get that u attains its minimum on  $\partial A$  or  $\partial B$ . WOLG, we assume that u attains its minimum on  $\partial A$ , then  $\frac{\partial u}{\partial n} < 0$  by the Hopf principle since u is not a constant. But this contradicts to  $\iint_{\partial A} \frac{\partial u}{\partial n} dS = Q > 0$ . So  $u \ge 0$  in D,
- (c) Suppose that there exists  $\mathbf{x}_0 \in D$  such that  $u(\mathbf{x}_0) = 0$ . Then by (b) and the Strong Minimum Principle, u is a constant in D. But this contradicts to  $\iint_{\partial A} \frac{\partial u}{\partial n} dS = Q > 0$ . So u > 0 in D. (Actually, we can prove that u > 0 on  $\partial A$  and  $\partial B$  by the Hopf principle again as in (b)).  $\Box$
- 7. (Extra Problem 1) Suppose  $u_1$  and  $u_2$  are solutions of the problem. Then

$$\Delta(u_1 - u_2) = u_1^3 - u_2^3 = (u_1 - u_2)(u_1^2 + u_1u_2 + u_2^2) \text{ in } D$$
$$\frac{\partial(u_1 - u_2)}{\partial n} + a(x)(u_1 - u_2) = 0 \text{ on } \partial D$$

Thus,

$$0 \leq \int_{D} (u_1 - u_2)^2 (u_1^2 + u_1 u_2 + u_2^2) = \int_{D} (u_1 - u_2) \Delta(u_1 - u_2)$$
  
= 
$$\int_{\partial D} (u_1 - u_2) \frac{\partial(u_1 - u_2)}{\partial n} - \int_{D} |Du_1 - Du_2|^2$$
  
= 
$$-\int_{\partial D} a(x)(u_1 - u_2)^2 - \int_{D} |Du_1 - Du_2|^2 \leq 0$$

since  $a(x) \ge 0$ .  $\Rightarrow \int_D (u_1 - u_2)^2 (u_1^2 + u_1 u_2 + u_2^2) = 0 \Rightarrow u_1 = u_2$  in D

8. (Extra Problem 2) (a)

$$E[u] := \int_D \frac{1}{2} (|Du|^2 + b(x)u^2) + f(x)u$$

(b) If u is a solution, then  $\forall v \in C_0^2(D), \ 0 = \int_D -v \triangle u + b(x)uv + f(x)v = \int_D Du \cdot Dv + b(x)uv + f(x)v \Rightarrow \forall w \in C^2(D), \ w = h \text{ on } \partial D$ , take v = w - u,

$$\begin{split} E[w] &= \int_D \frac{1}{2} |Dv|^2 + \frac{1}{2} |Du|^2 + Dv \cdot Du + \frac{1}{2} b(x)v^2 + \frac{1}{2} b(x)u^2 + b(x)uv + f(x)v + f(x)u \\ &= \int_D \frac{1}{2} |Dv|^2 + \frac{1}{2} b(x)v^2 + E[u] \\ &\geq E[u] \end{split}$$

since  $b(x) \ge 0$ .

Conversely,  $\forall v \in C_0^2(D), \forall t \in \mathbb{R}, \ w := u + tv \in C^2, \ w = u = h \ on \ \partial D$ 

$$\Rightarrow 0 = \frac{d}{dt} E[u + tv] \big|_{t=0} = \int_D Du \cdot Dv + b(x)uv + f(x)v$$
$$\Rightarrow \int_D v(-\Delta u + b(x)u + f(x)) = 0, \ \forall v \in C_0^2(D)$$
$$\Rightarrow -\Delta u + b(x)u + f(x) = 0 \ in \ D$$

## Exercise 7.2

2. Let  $r = |\mathbf{x}|$  and  $D_{\epsilon} := \{x | \epsilon < r < R\}$ , where R is large enough such that  $\phi = 0$  outside r < R/2. Using the Green's second identity,

$$\begin{split} &\iint_{D_{\epsilon}} \frac{1}{|\mathbf{x}|} \Delta \phi(\mathbf{x}) d\mathbf{x} = \iint_{\partial D_{\epsilon}} \left[ \frac{1}{|\mathbf{x}|} \cdot \frac{\partial \phi}{\partial n} - \phi \cdot \frac{\partial}{\partial n} \frac{1}{\mathbf{x}} \right] dS \\ &= \iint_{|\mathbf{x}|=R} \left[ \frac{1}{|\mathbf{x}|} \cdot \frac{\partial \phi}{\partial n} - \phi \cdot \frac{\partial}{\partial n} \frac{1}{\mathbf{x}} \right] dS + \iint_{|\mathbf{x}|=\epsilon} \left[ \frac{1}{|\mathbf{x}|} \cdot \frac{\partial \phi}{\partial n} - \phi \cdot \frac{\partial}{\partial n} \frac{1}{\mathbf{x}} \right] dS \\ &= -\iint_{|\mathbf{x}|=\epsilon} \left[ \frac{1}{\epsilon} \cdot \frac{\partial \phi}{\partial r} + \phi \cdot \frac{1}{c^2} \right] dS = -4\pi\overline{\phi} - 4\pi\epsilon\frac{\overline{\partial \phi}}{\partial r}, \end{split}$$

where  $\overline{\phi}$  denotes the average value of  $\phi$  on the sphere  $\{r = \epsilon\}$ , and  $\overline{\frac{\partial \phi}{\partial r}}$  denotes the average value of  $\frac{\partial \phi}{\partial r}$  on this sphere. Since  $\phi$  is continuous and  $\frac{\partial \phi}{\partial r}$  is bounded,

$$-4\pi\overline{\phi} - 4\pi\epsilon\overline{\partial\phi} \to -4\pi\phi(\mathbf{0}) \quad \text{as } \epsilon \to 0.$$

Hence,

$$\phi(\mathbf{0}) = -\iiint_{|\mathbf{x}| < R} \frac{1}{|\mathbf{x}|} \Delta \phi(\mathbf{x}) \frac{d\mathbf{x}}{4\pi}. \qquad \Box$$

3. Choosing  $D = B(\mathbf{x}_0, R)$  in the representation formula (1) and using the divergence theorem,

$$\begin{split} u(\mathbf{x}_{0}) &= \iint_{\partial B(\mathbf{x}_{0},R)} \left[ -u(\mathbf{x}) \cdot \frac{\partial}{\partial n} \left( \frac{1}{|\mathbf{x} - \mathbf{x}_{0}|} \right) + \frac{1}{|\mathbf{x} - \mathbf{x}_{0}|} \cdot \frac{\partial u}{\partial n} \right] \frac{dS}{4\pi} \\ &= \iint_{|\mathbf{x} - \mathbf{x}_{0}| = R} \left[ \frac{1}{R^{2}} u(\mathbf{x}) + \frac{1}{R} \frac{\partial u}{\partial n} \right] \frac{dS}{4\pi} \\ &= \frac{1}{4\pi R^{2}} \iint_{|\mathbf{x} - \mathbf{x}_{0}| = R} u dS + \frac{1}{4\pi R} \iiint_{|\mathbf{x} - \mathbf{x}_{0}| < R} \Delta u d\mathbf{x} \\ &= \frac{1}{4\pi R^{2}} \iint_{|\mathbf{x} - \mathbf{x}_{0}| = R} u dS. \quad \Box \end{split}$$

4. (Extra Problem)  $\forall \phi(x) \in C_c^2(\mathbb{R}^2)$ , we need to show that

$$\phi(0) = \int \int_{\mathbb{R}^2} \log |x| \triangle \phi(x) \frac{dx}{2\pi}$$

proof: Let r = |x| and  $D_{\varepsilon} := x : \varepsilon < r < R$ , where R is large enough such that  $\phi = 0$  if  $r \ge R/2$ . Using the Green's identity,

$$\int \int_{D_{\varepsilon}} \log |x| \Delta \phi(x) dx = \int_{\partial D_{\varepsilon}} [\log |x| \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial} \log |x|] dS$$
$$= \int_{|x|=\varepsilon} [\log |x| \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial} \log |x|] dS$$
$$= -\int_{|x|=\varepsilon} [\log \varepsilon \frac{\partial \phi}{\partial r} - \phi \frac{1}{\varepsilon}] dS$$
$$= 2\pi \bar{\phi} - 2\pi \varepsilon \log \varepsilon \frac{\bar{\partial} \phi}{\partial r}$$

where  $\bar{\phi}$ ,  $\frac{\bar{\partial}\phi}{\partial r}$  denote the average value of  $\phi$ ,  $\frac{\partial\phi}{\partial r}$  on the sphere  $|x| = \varepsilon$ . Since  $\phi$  is continuous and  $\frac{\partial\phi}{\partial r}$  is bounded,

$$2\pi\bar{\phi} - 2\pi\varepsilon\log\varepsilon\frac{\partial\phi}{\partial r} \to 2\pi\phi(0), \ as \ \varepsilon \to 0$$

### Exercise 7.3

1. Suppose  $G_1$  and  $G_2$  both are the Green's functions for the operator  $-\Delta$  and the domain D at the point  $\mathbf{x}_0 \in d$ . By (i) and (iii), we obtain that  $G_1(\mathbf{x}) + \frac{1}{4\pi |\mathbf{x} - \mathbf{x}_0|}$  and  $G_2(\mathbf{x}) + \frac{1}{4\pi |\mathbf{x} - \mathbf{x}_0|}$  both are harmonic functions in D. Thus, by (ii) and the uniqueness theorem of harmonic function, we have

$$G_1(\mathbf{x}) + \frac{1}{4\pi |\mathbf{x} - \mathbf{x}_0|} = G_2(\mathbf{x}) + \frac{1}{4\pi |\mathbf{x} - \mathbf{x}_0|},$$

that is, the Green's function is unique.  $\hfill \Box$ 

3. In the textbook,

$$A_{\epsilon} = \iint_{|\mathbf{x}-\mathbf{a}|=\epsilon} (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})dS,$$

where  $u(\mathbf{x}) = G(\mathbf{x}, \mathbf{a})$  and  $v(\mathbf{x}) = G(\mathbf{x}, \mathbf{b})$ . Note that

$$u(\mathbf{x}) = G(\mathbf{x}, \mathbf{a}) = -\frac{1}{4\pi |\mathbf{x} - \mathbf{a}|} + H(\mathbf{x}, \mathbf{a})$$

, where  $H(\mathbf{x}, \mathbf{a})$  is a harmonic function throughout the domain *D*. What's more,  $v(\mathbf{x}) = G(\mathbf{x}, \mathbf{b})$  is a harmonic function in  $\{|\mathbf{x} - \mathbf{a}| < \epsilon\}$ , then we have

$$\begin{split} A_{\epsilon} &= \iint_{|\mathbf{x}-\mathbf{a}|=\epsilon} \left[ (-\frac{1}{4\pi |\mathbf{x}-\mathbf{a}|} + H(\mathbf{x},\mathbf{a})) \frac{\partial v}{\partial n} - v \frac{\partial}{\partial n} (-\frac{1}{4\pi |\mathbf{x}-\mathbf{a}|} + H(\mathbf{x},\mathbf{a})) \right] dS \\ &= \iint_{|\mathbf{x}-\mathbf{a}|=\epsilon} \left[ -\frac{1}{4\pi |\mathbf{x}-\mathbf{a}|} \frac{\partial v}{\partial n} + v \frac{\partial}{\partial n} (\frac{1}{4\pi |\mathbf{x}-\mathbf{a}|}) \right] dS \\ &+ \iint_{|\mathbf{x}-\mathbf{a}|=\epsilon} \left[ H(\mathbf{x},\mathbf{a}) \frac{\partial v}{\partial n} - v \frac{\partial}{\partial n} H(\mathbf{x},\mathbf{a}) \right] dS \\ &= v(\mathbf{a}) - \iiint_{|\mathbf{x}-\mathbf{a}|<\epsilon} \left[ H(\mathbf{x},\mathbf{a}) \Delta v - v \Delta H(\mathbf{x},\mathbf{a}) \right] d\mathbf{x} = v(\mathbf{a}). \end{split}$$

Here we use the representation formula (7.2.1) and  $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$ . So  $\lim_{\epsilon \to 0} A_{\epsilon} = v(\mathbf{a}) = G(\mathbf{a}, \mathbf{b})$ .

#### Exercise 7.4

1. By (i), we know that G(x) is a linear function in  $[0, x_0]$  and  $[x_0, l]$ . Since (ii) and G(x) is continuous at  $x_0$ , we have

$$G(x) = \begin{cases} kx, & 0 < x \le x_0; \\ \frac{kx_0}{x_0 - l}(x - l), & x_0 < x < l. \end{cases}$$

Hence,  $G(x) + \frac{1}{2}|x - x_0|$  is harmonic at  $x_0$  if and only if

$$\left(G(x) + \frac{1}{2}|x - x_0|\right)'\Big|_{x_0^-} = \left(G(x) + \frac{1}{2}|x - x_0|\right)'\Big|_{x_0^+},$$

if and only if

$$k = \frac{l - x_0}{l}.$$

So the one-dimensional Green's function for the interval (0, l) at the point  $x_0 \in (0, l)$  is

$$G(x) = \begin{cases} \frac{l - x_0}{l} x, & 0 < x \le x_0; \\ -\frac{x_0}{l} (x - l), & x_0 < x < l. \end{cases}$$

2. Assume that h(x, y) is a continuous function that vanished outside  $\{(x, y)|x^2+y^2 \leq R^2\}$  and  $|h(x, y)| \leq M$ . Then by the formula (3), we have

$$\begin{aligned} |u(x_0, y_0, z_0)| &\leq \frac{M}{2\pi} \iint_{\{(x,y)|x^2 + y^2 \leq R^2\}} [(x - x_0)^2 + (y - y_0)^2 + z_0^2]^{-\frac{3}{2}} dx dy \\ &\leq \frac{MR^2}{2(\sqrt{x_0^2 + y_0^2 + z_0^2} - R)^3}, \quad \text{when } \sqrt{x_0^2 + y_0^2 + z_0^2} > R. \end{aligned}$$

Therefore, u satisfies the condition at infinity:

$$u(\mathbf{x}) \to 0, \quad \text{as } |\mathbf{x}| \to \infty. \quad \Box$$

3. From (3), we have

$$u(x_0, y_0, z_0) = \frac{z_0}{2\pi} \iint [(x - x_0)^2 + (y - y_0)^2 + z_0^2]^{-\frac{3}{2}} h(x, y) dx dy$$
$$= \frac{z_0}{2\pi} \iint [x^2 + y^2 + z_0^2]^{-\frac{3}{2}} h(x + x_0, y + y_0) dx dy.$$

Now we change variables such that  $x = z_0 s \cos \theta$ ,  $y = z_0 s \sin \theta$ . Then

$$u(x_0, y_0, z_0) = \frac{z_0}{2\pi} \int_0^{2\pi} \int_0^\infty (z_0^2 s^2 + z_0^2)^{-\frac{3}{2}} h(z_0 s \cos \theta + x_0, z_0 s \sin \theta + y_0) z_0^2 s \, ds d\theta$$
  
=  $\frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty s(s^2 + 1)^{-\frac{3}{2}} h(z_0 s \cos \theta + x_0, z_0 s \sin \theta + y_0) \, ds d\theta.$ 

Since

$$\int_0^\infty s(s^2+1)^{-\frac{3}{2}} ds = -(s^2+1)^{-\frac{3}{2}} \Big|_0^\infty = 1,$$
$$\lim_{z_0 \to 0} u(x_0, y_0, z_0) = h(x_0, y_0),$$

if the limit can be taken in the integration, for example, when h(x, y) is bounded.

5. Here is one of suggested explains.

Since the half-plane  $\{y > 0\}$  is not bounded, this only means that the solution is not unique under the (uncompleted) boundary condition or it is not a well-posed problem. But generally it will be a well-posed problem if we add another boundary condition, such as

$$u(\mathbf{x}) \to 0, \quad \text{as } |\mathbf{x}| \to \infty. \quad \Box$$

6. (a) Using the method of reflection, as in the dimension three, we have

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x}_0| - \frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x}_0^*|,$$

where  $\mathbf{x} = (x, y), \mathbf{x}_0 = (x_0, y_0), \mathbf{x}_0^* = (x_0, -y_0).$ 

(b) Since

$$-\frac{\partial G}{\partial y} = \frac{y + y_0}{2\pi[(x - x_0)^2 + (y + y_0)^2]} - \frac{y - y_0}{2\pi[(x - x_0)^2 + (y - y_0)^2]}$$
$$= \frac{y_0}{\pi[(x - x_0)^2 + y_0^2]}$$

on y = 0, the solution is given by

$$u(x_0, y_0) = \frac{y_0}{\pi} \int \frac{h(x)}{(x - x_0)^2 + y_0^2} dx$$

(c) By (b), we have

$$u(x_0, y_0) = \frac{y_0}{\pi} \int \frac{h(x)}{(x - x_0)^2 + y_0^2} dx = \frac{1}{\pi} \int \frac{1}{x^2 + 1} dx = 1. \qquad \Box$$

7. (a) If  $u(x,y) = f(\frac{x}{y})$ , then

$$u_x = \frac{1}{y} f'(\frac{x}{y}), \ u_y = -\frac{x}{y^2} f'(\frac{x}{y}).$$
$$u_{xx} + u_{yy} = \frac{1}{y^2} f''(\frac{x}{y}) + \frac{x^2}{y^4} f''(\frac{x}{y}) + \frac{2x}{y^3} f'(\frac{x}{y})$$

So f has to satisfy the following ODE

$$(1+t^2)f''(t) + 2tf'(t) = 0.$$
$$f(t) = \int_0^t \frac{a}{1+s^2} ds + b = a \arctan t + b,$$

where a, b are contants.

(b)

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{1}{y}f'(\frac{x}{y})\frac{x}{r} - \frac{x}{y^2}f'(\frac{x}{y})\frac{y}{r} = 0.$$

(c) Using the polar coordinate,

$$v(r,\theta) = c\theta + d,$$

where c, d are constants. So in the (x, y)-coordinate, we have

$$v(x,y) = c \arctan \frac{y}{x} + d.$$

(d) By (a),

$$u(x,y) = f(\frac{x}{y}) = a \arctan \frac{x}{y} + b.$$

 $\operatorname{So}$ 

$$h(x) = \lim_{y \to 0} u(x, y) = \begin{cases} \frac{\pi}{2}a + b, & x > 0; \\ b, & x = 0; \\ -\frac{\pi}{2}a + b, & x < 0. \end{cases}$$

(e) By (d), we see that h(x) is not continuous unless u is a constant. This agrees with the condition that h(x) is continuous in Exercise 6.  $\Box$ 

9. Here the method of reflection is used. Let  $\mathbf{x}_0 = (x_0, y_0, z_0)$  such that  $ax_0 + by_0 + cz_0 > 0$ , then its reflection point  $\mathbf{x}'_0 = (x'_0, y'_0, z'_0)$  about the hyperplane  $\{ax + by + cz = 0\}$  satisfies

$$\begin{aligned} x_0' &= x_0 - \frac{2a}{a^2 + b^2 + c^2} (ax_0 + by_0 + cz_0), \\ y_0' &= y_0 - \frac{2b}{a^2 + b^2 + c^2} (ax_0 + by_0 + cz_0), \\ z_0' &= z_0 - \frac{2c}{a^2 + b^2 + c^2} (ax_0 + by_0 + cz_0). \end{aligned}$$

So the Green's function for the tilted half-space  $\{(x, y, z): ax + by + cz > 0\}$  is given by

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi |\mathbf{x} - \mathbf{x}_0|} + \frac{1}{4\pi |\mathbf{x} - \mathbf{x}_0'|}.$$

10. In case  $\mathbf{x}_0$ , the formula for the Green's function is

$$G(\mathbf{x}, \mathbf{0}) = -\frac{1}{4\pi |\mathbf{x}|} + \frac{1}{4\pi a}$$

since  $-\frac{1}{4\pi|\mathbf{x}|} + \frac{1}{4\pi a} \in C^2$  and is harmonic in  $B_a(\mathbf{0})$ , except at the point  $\mathbf{x} = \mathbf{0}$ ;  $\left[-\frac{1}{4\pi|\mathbf{x}|} + \frac{1}{4\pi a}\right]\Big|_{|\mathbf{x}|=a} = 0$ ; and  $\left[-\frac{1}{4\pi|\mathbf{x}|} + \frac{1}{4\pi a}\right] + \frac{1}{4\pi|\mathbf{x}|}$  is harmonic at  $\mathbf{0}$ .

13. Let  $\mathbf{x}_0 \in D$ , then we have

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi |\mathbf{x} - \mathbf{x}_0|} + \frac{1}{4\pi |r_0 \mathbf{x}/a - a\mathbf{x}_0/r_0|}$$

is the Green's function at  $\mathbf{x}_0$  for the whole ball, where *a* is the radius and  $r_0 = |\mathbf{x}_0|$ . Let  $\mathbf{x}'_0 = (x_0, y_0, -z_0)$ , then

$$G(\mathbf{x}, \mathbf{x}'_0) = -\frac{1}{4\pi |\mathbf{x} - \mathbf{x}'_0|} + \frac{1}{4\pi |r_0 \mathbf{x}/a - a\mathbf{x}'_0/r_0|}$$

is the Green's function at  $\mathbf{x}'_0$  for the whole ball. Note that  $G(\mathbf{x}, \mathbf{x}_0) = G(\mathbf{x}, \mathbf{x}'_0)$  on  $\partial D$  and  $G(\mathbf{x}, \mathbf{x}'_0)$  is a harmonic function on D. Hence,  $G(\mathbf{x}, \mathbf{x}_0) - G(\mathbf{x}, \mathbf{x}'_0)$  is the Green's function for D by the uniqueness of the Green's function.  $\Box$ 

15. (a) If v(x,y) is harmonic and  $u(x,y) = v(x^2 - y^2, 2xy)$ , then

$$u_x = 2xv_x + 2yv_y, \qquad u_y = -2yv_x + 2xv_y.$$

Thus

$$u_{xx} + u_{yy} = 2v_x + 4x^2v_{xx} + 4xyv_{xy} + 4xyv_{yx} + 4y^2v_{yy} - 2v_x + 4y^2v_{xx} - 4xyv_{xy} - 4xyv_{yx} + 4x^2v_{yy} = 4(x^2 + y^2)(v_{xx} + v_{yy}) = 0.$$

(b) Using the polar coordinates, the transformation si

$$(r\cos\theta, r\sin\theta) \mapsto (r^2\cos 2\theta, r^2\sin 2\theta)$$

Therefore, it maps the first quadrant onto the half-plane  $\{y > 0\}$ . Note that the transformation is one-one which also maps the *x*-positive-axis to *x*-positive-axis and *y*-positive-axis to *x*-negative-axis.  $\Box$ 

17. (a) Here the method of reflection is used and you can also use the result of Exercise 155 to find the Green's function for the quadrant Q. Let  $\mathbf{x}_0 = (x_0, y_0) \in Q$ ,  $\mathbf{x}_0^y = (-x_0, y_0)$ ,  $\mathbf{x}_0^x = (x_0, -y_0)$  and  $\mathbf{x}_0^0 = (-x_0, -y_0)$ . Then it is easy to verify that

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x}_0| - \frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x}_0^y| - \frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x}_0^x| + \frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x}_0^0|$$

is the Green's function at  $\mathbf{x}_0$  for the quadrant Q.

(b) (Extra Problem 3)

$$\begin{split} \frac{\partial G}{\partial n}\Big|_{x=0,y>0} &= -\frac{\partial G}{\partial x} = \frac{x_0}{\pi} \left(\frac{1}{|\mathbf{x} - \mathbf{x}_0|^2} - \frac{1}{|\mathbf{x} - \mathbf{x}_0^0|^2}\right) \\ \frac{\partial G}{\partial n}\Big|_{x>0,y=0} &= -\frac{\partial G}{\partial y} = \frac{y_0}{\pi} \left(\frac{1}{|\mathbf{x} - \mathbf{x}_0|^2} - \frac{1}{|\mathbf{x} - \mathbf{x}_0^0|^2}\right) \end{split}$$

By formula (1) in Section 7.3 in the textbook,

$$\begin{split} u(\mathbf{x}_{0}) &= \iint_{\text{bdy } D} u(\mathbf{x}) \frac{\partial G(\mathbf{x}, \mathbf{x}_{0})}{\partial n} dS \\ &= \frac{x_{0}}{\pi} \int_{0}^{\infty} g(y) (\frac{1}{|\mathbf{x} - \mathbf{x}_{0}|^{2}} - \frac{1}{|\mathbf{x} - \mathbf{x}_{0}^{0}|^{2}}) dy + \frac{y_{0}}{\pi} \int_{0}^{\infty} h(x) (\frac{1}{|\mathbf{x} - \mathbf{x}_{0}|^{2}} - \frac{1}{|\mathbf{x} - \mathbf{x}_{0}^{0}|^{2}}) dx \quad \Box \end{split}$$

- 19. Consider the four-dimensional laplacian  $\Delta u = u_{xx} + u_{yy} + u_{zz} + u_{ww}$ . Show that its fundamental solution is  $r^{-2}$ , where  $r^2 = x^2 + y^2 + z^2 + w^2$ . proof: Direct computation according to its definition.
- 20. Using the conclusion in 17 and 20, we have the singular part of the Green function is  $-\frac{1}{8\omega_4}|\mathbf{x}-\mathbf{x}_0|^{-2}$ , where  $\omega_4$  is the volume of  $S^3$ ,  $\mathbf{x} = (x, y, z, w)$  and  $\mathbf{x}_0 = (x_0, y_0, z_0, w_0)$ . So the Green function is

$$-\frac{1}{8\omega_4}|\mathbf{x}-\mathbf{x}_0|^{-2}+\frac{1}{8\omega_4}|\mathbf{x}-\mathbf{x}_0^-|^{-2},$$

where  $\mathbf{x}_0^- = (x_0, y_0, z_0, -w_0).$ 

21. By the formula (2)(3) in Section 7.3, we get

$$u(\mathbf{x}_0) = \int \int_{bdyD} (u\frac{\partial N}{\partial n} - \frac{\partial u}{\partial n}N) = \int \int_{bdyD} -\frac{\partial u}{\partial n}N$$

since  $\frac{\partial N}{\partial n} = 0$  on the boundary. Thus we proved the following theorem:

If  $N(\mathbf{x}, \mathbf{x}_0)$  is the Neumann function, then the solution of the Neumann problem is given by the formula

$$u(\mathbf{x}_0) = -\int \int_{bdyD} \frac{\partial u}{\partial n} N$$

22. (Extra Problem 4)By the formula (4) in Section7.4,

$$u(\mathbf{x}_0) = \frac{z_0}{2\pi} \int \int_{z=0}^{z_0} \frac{\sqrt{x^2 + y^2 + 1}}{|\mathbf{x} - \mathbf{x}_0|^3} dS.$$

If the bounded condition is dropped, then we have v = u + az where a is arbitrary real number is also a solution.