MATH 4050 Real Analysis

Tutorial 13 (April 19, 21)

The following were discussed in the tutorial this week.

1. Let f be defined by
$$f(x) = \begin{cases} \sin x & \text{if } \sin x \in \mathbb{Q}, \\ \cos^2 x & \text{if } \sin x \notin \mathbb{Q}. \end{cases}$$
Is f measurable/integrable over the interval $[0, \frac{\pi}{2}]$? If yes, find the value of $\displaystyle\int_0^{\frac{\pi}{2}} f$; if not explain why. Provide your reasoning.

2. Let $1 \le p < \infty$. Suppose $E \subseteq \mathbb{R}$ is measurable. If $f, f_n \in L_p(E)$ $(\forall n \in \mathbb{N})$ are such that $\lim_n \|f_n\|_p = \|f\|_p$ and $\{f_n\}$ converges to f almost everywhere on E, show that
$$\lim_n \|f_n - f\|_p = 0.$$
Does the same result hold if $p = \infty$?

3. (Fubini) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of nondecreasing (or nonincreasing) real-valued functions on an interval $[a, b]$ such that $\sum_{n=1}^{\infty} f_n(x) = s(x)$ exists and is finite in $[a, b]$. Show that
$$s'(x) = \sum_{n=1}^{\infty} f_n'(x) \qquad \text{a.e. in } [a, b].$$

4. Prove the following version of Vitali lemma:

   Suppose $m^*(E) < \infty$. Let $\mathcal{U}$ be a Vitali cover of E. Then there is a countable disjoint subcollection $\{I_j\} \subseteq \mathcal{U}$ such that
$$m^*(E \setminus \bigcup_j I_j) = 0.$$
The assumption "$m^*(E) < \infty$" can also be removed.

5. Let E be a (not necessarily countable) union of a family of quite arbitrary intervals, each being open, closed, half open and half closed. Prove that E is Lebesgue measurable.