MATH 2550

L3.

Keywords: Line Integral.

Tools needed to talk about line integrals:

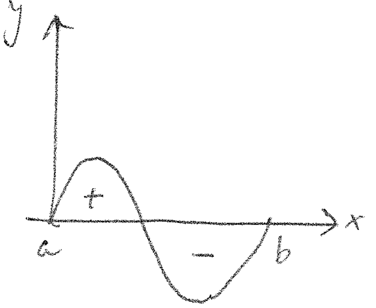
(I) concept of $\int_a^b f(x)\,dx$

(II) concept of ds (i.e. infinitesimal length of a curve C)

(III) concept of tangent vectors to C.

Quick Review of $\int_a^b f(x)\,dx$

(i) $\int_a^b f(x)\,dx$ = "signed" area "below" the ~~function~~ curve $y=f(x)$; $a \le x \le b$.



The word "signed" means "having + or − signs".

(ii) To obtain the number $\int_a^b f(x)\,dx$, we do the following:

(a) We have a domain of integration, i.e. $[a,b]$,

(b) We partition $[a,b]$ into n subintervals, $[a,x_1]$, $[x_1,x_2]$, ... $[x_{i-1},x_i]$, ... $[x_{n-1},b]$.

(c) On each of these subintervals, we draw a rectangle of height $f(\xi_i)$, where ξ_i is any convenient point in $[x_{i-1},x_i]$

(d) We form the sum $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$, where Δx_i = width of $[x_{i-1},x_i]$.

(e) We let $n \to \infty$ and study the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta x_i \quad \text{which equals } \int_a^b f(x)\,dx$$

E.g. $f(x) = x$

$[a,b] = [1, 2]$.

Let's partition $[a,b]$ by letting $x_0 = 1$, $x_1 = 1+\frac{1}{n}$, $x_2 = 1+\frac{2}{n}$,

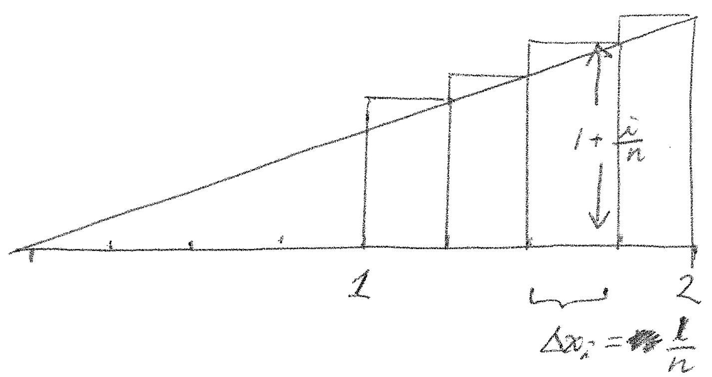
$\cdots$ $x_i = 1+\frac{i}{n}$, $\cdots$, $x_n = 1+\frac{n}{n} = 2$.

Then we have $\Delta x_i \overset{\text{def.}}{=} x_i - x_{i-1} = \frac{1}{n}$

Let's choose ξ_i to be the right end-point of the subinterval $[x_{i-1}, x_i]$. Then we get

$$\xi_i = x_i = 1+\frac{i}{n}$$

Now we get a rectangle with width $\frac{1}{n}$ and height $1+\frac{i}{n}$.



The "approximate" area is the sum given by

$$\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)\frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)$$

$$= \frac{1}{n}\left[n + \frac{1}{\cancel{n}}\left(\frac{\cancel{n}(n+1)}{2}\right)\right] = \frac{1}{n}\left[\frac{3n+1}{2}\right]$$

As $n\to\infty$, we get $\lim\limits_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)\frac{1}{n} = \lim\limits_{n\to\infty}\left(\frac{3}{2}+\frac{1}{2n}\right) = \frac{3}{2}$

$\therefore \int_1^2 x\,dx = \frac{3}{2}$

Integration over Curves

Next, we extend the previous computational steps to the case when $[a,b]$ is replaced by a curve C, when C is given by a function (vector-valued) $\cancel{\phi}$

$$\begin{array}{ccc} \vec{\alpha} : [a,b] & \longrightarrow & \mathbb{R}^{2,3} \\ \Psi & & \Psi \\ t & \longmapsto & \vec{\alpha}(t) \end{array}$$

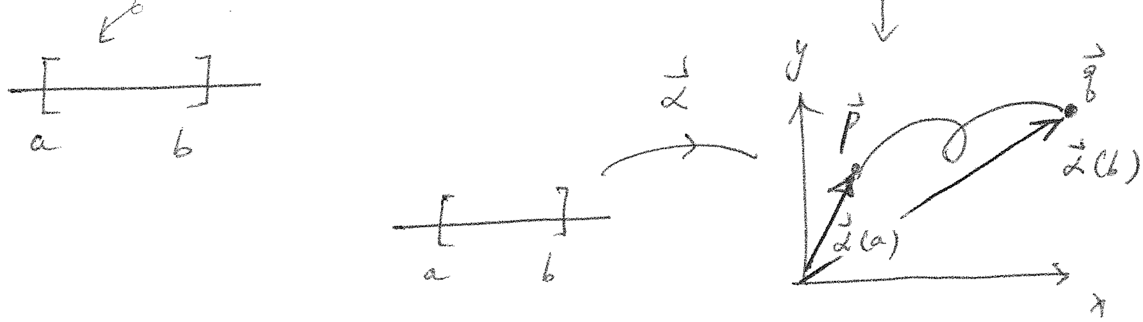
RMK: I use the same notation $[a,b]$, but you can also use any other such as $[c,d]$, $[p,q]$.

For simplicity, let's consider curve in $\mathbb{R}^2$, therefore we have

$$\begin{array}{ccc} \vec{\alpha} : [a,b] & \longrightarrow & \mathbb{R}^{2} \\ \Psi & & \Psi \\ t & \longmapsto & (x(t), y(t)) \end{array}$$

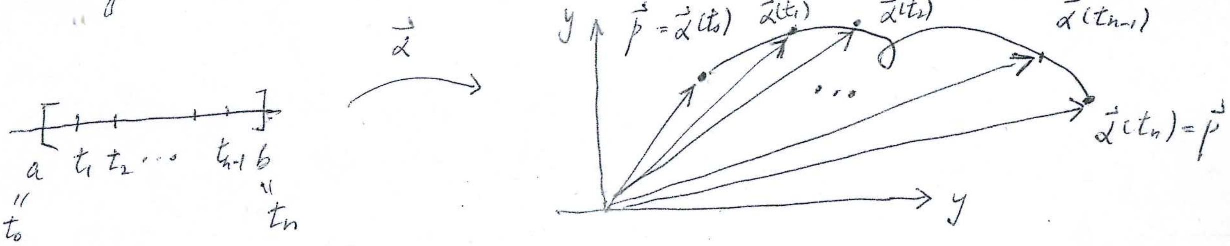
Now we repeat the things we have done before to define $\int_a^b f(x)\,dx$,

Step 1) $[a,b]$ which is now replaced by C, α



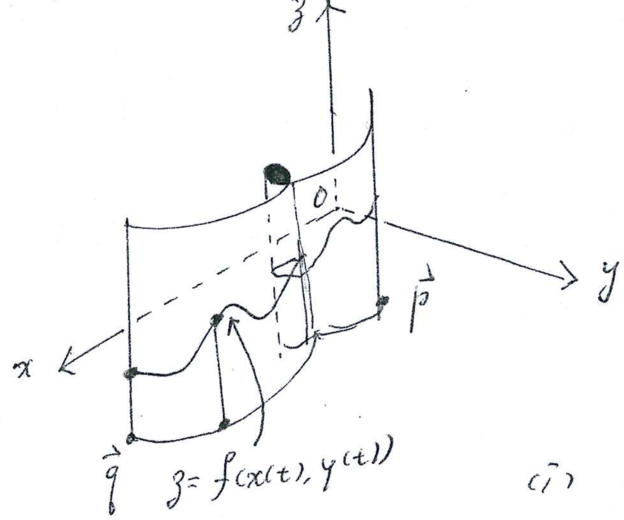
Step 2) We partition C by partitioning $[a,b]$

Partitioning $[a, b]$ into n subintervals gives rise to n sub-arcs in C.



Let $f(x, y)$ be a function defined on the entire xy-plane.

We consider $f(\vec{\alpha}(t)) = f(x(t), y(t)) = f$ "restricted" to the curve C.

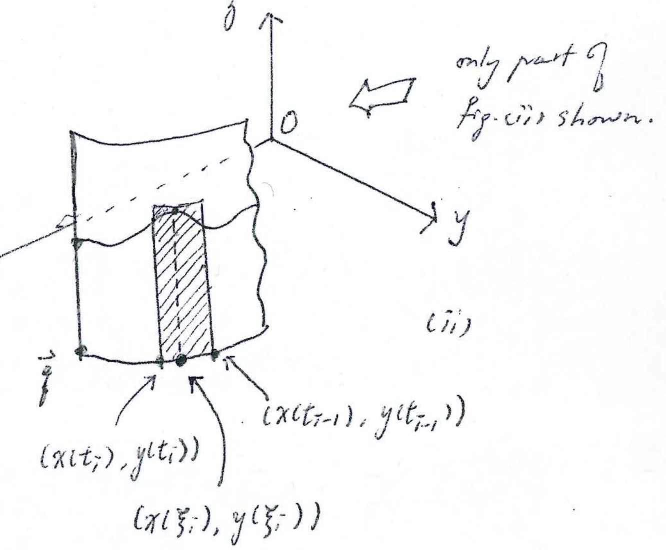


(i)

Then we can form rectangles (actually "curved" or curvilinear rectangles) such as the shaded one here.

Here ξ_i is any convenient point in the subinterval $[t_{i-1}, t_i]$.

Now, we need to approximate the "arc" $(x(t_{i-1}), y(t_i))\ (x(t_i), y(t_i))$ (or in vector notation $\overset{\frown}{\vec{\alpha}(t_{i-1})\ \vec{\alpha}(t_i)}$) by a straight line segment joining $(x(t_{i-1}), y(t_{i-1}))$ to $(x(t_i), y(t_i))$.



(ii)

Since we're interested in the arc-length, its approximation is:

$$\sqrt{(x(t_i)-x(t_{i-1}))^2+(y(t_i)-y(t_{i-1}))^2}$$

denoted by the symbol ΔS_i (meaning approximation of the i^{th} arc-length).
Using this, we obtain approximation of the i^{th} curvilinear rectangle by

Area of the i^{th} curvilinear rectangle $=$

$$f(x(\xi_i),y(\xi_i))\cdot\Delta S_i =$$

$$f(x(\xi_i);y(\xi_i))\cdot\sqrt{(\Delta x_i)^2+(\Delta y_i)^2}$$

where $\Delta x_i \overset{\text{def.}}{=} x_i - x_{i-1}$

$\Delta y_i \overset{\text{def.}}{=} y_i - y_{i-1}$

<u>Step 3)</u> The sum of the areas of these n curvilinear rectangles is equal to (approximately):

$$\sum_{i=1}^{n} f(x(\xi_i),y(\xi_i))\sqrt{(\Delta x_i)^2+(\Delta y_i)^2}$$

$$=\sum_{i=1}^{n} f(x(\xi_i),y(\xi_i))\sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2+\left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\,|\Delta t_i|$$

$$\cancel{=\sum_{i=1}^{n} f(x(\xi_i),y(\xi_i))\sqrt{1+\left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\,\Delta t_i}$$

, since $\Delta t_i = t_i - t_{i-1}$ & $t_i > t_{i-1}$.

$$=\sum_{i=1}^{n} f(x(\xi_i),y(\xi_i))\sqrt{1+\frac{\Delta y_i}{\Delta t_i}}\quad\Delta t_i$$

<u>Step 4)</u> Letting $n\to\infty$, the sum

$$\sum_{i=1}^{n} f(x(\xi_i),y(\xi_i))\sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2+\left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\;\Delta t_i$$

approaches the number denoted by

$$\int_{t=a}^{t=b} f(x(t),y(t))\underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt}.$$

this term is denoted by ds, i.e. the infinitesimal arc-length

E.g. $\vec{\alpha}(t) = (\cos t, \sin t) \quad 0 \le t \le 2\pi$

$f(x,y) = h$, where h is a constant number.

Then $\vec{\alpha} : [0, 2\pi] \longrightarrow \mathbb{R}^2$ gives rise to a curve C, which

$\quad\quad t \longmapsto (\cos t, \sin t)$ is a circle of radius 1, centered at the Origin.

<u>Q</u>: Compute $\int_C f \, ds$.

<u>A</u>: First of all, we remark that

$\int_C f \, ds$ is the short form for

$$\int_{t=a}^{t=b} f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt .$$

Now $f(x,y) = k$ for any point (x,y) in the plane, therefore when restricted to the curve given by $(x(t), y(t))$, where

$x(t) = \cos t, \; y(t) = \sin t$,

$f(\cos t, \sin t) = h$ also.

$$\text{Hence } \int_C f \, dS = \int_{t=0}^{t=2\pi} h \cdot \underbrace{\sqrt{\left(\frac{d\cos t}{dt}\right)^2 + \left(\frac{d\sin t}{dt}\right)^2}}_{=1} \, dt$$

$$= 2\pi h.$$

which is just the surface area of the cylinder of radius 1 & height h.

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<u>RMK</u> $\int_C f \, ds$ is called "line integral of the 1st kind" or "integral of scalar function on a curve."