MATH 2550

L1

Vectors

A vector is a mathematical object with both (i) direction & (ii) magnitude

* Geometrically, it is represented by an arrow, which has (i) & (ii).

In this course, we're interested mainly in vectors in the 2D plane (i.e. $\mathbb{R}^2$) or 3D space (i.e. $\mathbb{R}^3$).

Abbreviation: We'll write $\vec{v} \in \mathbb{R}^{2,3}$ to mean "$\vec{v}$ is a vector in the 2D plane or the 3D space.".

* $\mathbb{R}^2$.

Vectors in $\mathbb{R}^2$ can be represented by 2 numbers, written in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \text{ or } (x, y) \text{ or } x\hat{i} + y\hat{j} \quad \left(\text{here } \hat{i} \stackrel{\text{def.}}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \hat{j} \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

RMK * In books on "Linear Algebra", the notation $\begin{pmatrix} x \\ y \end{pmatrix}$ is often used.

* In books on "Coordinated Geometry" or "Analytic Geometry", the notation (x, y) is often used.

* We'll use ALL three notations interchangeably,

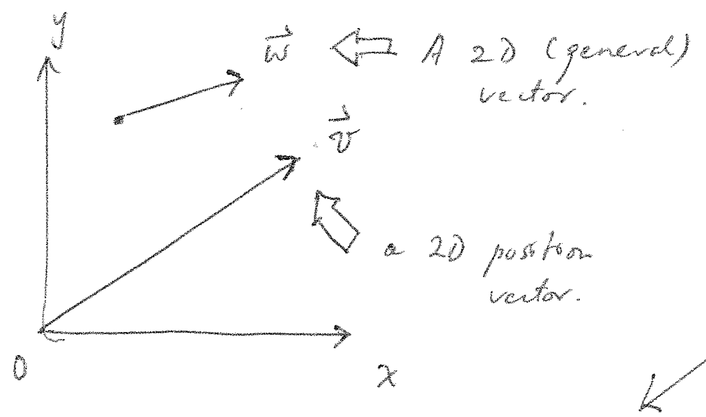
* Magnitude is also known as "length" or "norm".

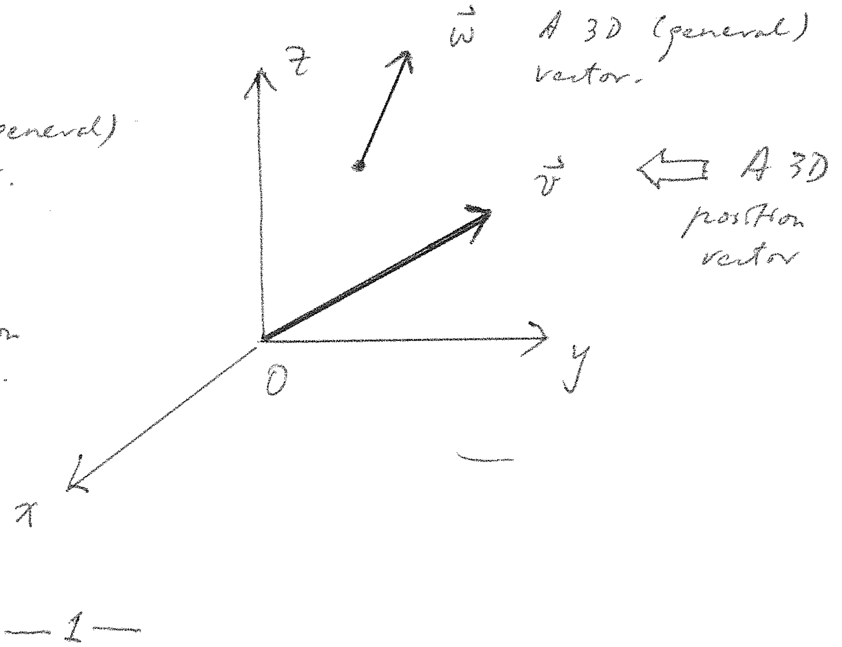
Properties of Vectors

There are 2 kinds of vectors

* position vectors (this kind of vectors have their tails at the origin).

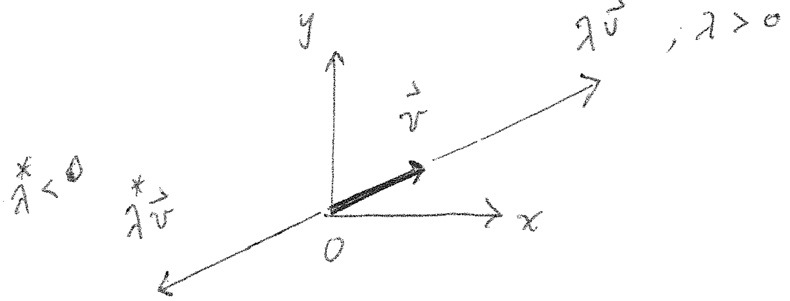
* general vectors



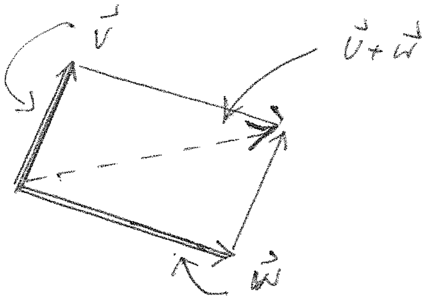


(Equality of Vectors) Two vectors are "equal" iff (= if and only if) they have the same <u>direction</u> & <u>magnitude</u>.

(Scalar multiplication of Vectors) given a vector $\vec{v}$, we can "scale" it. Such scaling makes the vector longer or shorter, or points in the opposite direction. In picture, it is



(Addition of Vectors) given two vectors $\vec{v}$ & $\vec{w}$, their "sum" is also a vector.



(Details of Vector Arithmetic)

given $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\vec{w} = \begin{pmatrix} p \\ q \end{pmatrix}$, then

$$\lambda\vec{v} = \lambda\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} \qquad \text{scalar multiplication}$$

$$\vec{v}+\vec{w} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} x+p \\ y+q \end{pmatrix} \qquad \text{addition.}$$

(Products of Two Vectors)

There are 2 kinds of products, viz. (i) dot product
(ii) cross product.

(Dot Product)

Given two vectors $\vec{v}$ & $\vec{w}$ in $\mathbb{R}^{2,3}$, the dot product $\vec{v}\cdot\vec{w}$ is the number defined by

$$\vec{v}\cdot\vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta \quad \text{———} \quad (*)$$

where $\|\vec{v}\|$ = magnitude of $\vec{v}$
$\|\vec{w}\|$ = magnitude of $\vec{w}$
and $\cos\theta$ = cosine of the (smaller) angle between $\vec{v}$ & $\vec{w}$.

(Alternative Computation)

One can show that $\vec{v}\cdot\vec{w} = \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix} = xp + yq$

if $\vec{v}, \vec{w} \in \mathbb{R}^2$.

Similarly, if $\vec{v}, \vec{w} \in \mathbb{R}^3$, $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\vec{w} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$, then

$$\vec{v}\cdot\vec{w} = xp + yq + zr.$$

(Use of $(*)$)

Often times, $(*)$ can be used to compute $\cos\theta$.

(Special Case — Orthogonal vectors)

Note that $\cos\theta = 0$ iff $\theta = \pi/2$, so $(*)$ implies that for <u>non-zero</u> vectors (i.e. magnitudes are non-zero),

$$\vec{v}\cdot\vec{w} = 0 \quad \text{iff} \quad \theta = \pi/2$$

<u>Notation</u>: If $\theta = \pi/2$, we say "$\vec{v}$ is orthogonal to $\vec{w}$" & denote this by $\vec{v} \perp \vec{w}$.

(Cross Product)

Given two vectors in $\mathbb{R}^3$, say $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\vec{w} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$, one can compute a vector "orthogonal" to $\vec{v}$ & $\vec{w}$ & having magnitude $\|\vec{v}\|\|\vec{w}\| \sin\theta$ (θ is the angle between $\vec{v}$ & $\vec{w}$). by

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p & q & r \end{vmatrix}$$

$$= (yr - qz)\hat{i} + (pz - xr)\hat{j} + (qx - yp)\hat{k}$$

RMK: cross product can only be defined for certain dimensions, e.g. $\mathbb{R}^3$.

Curves

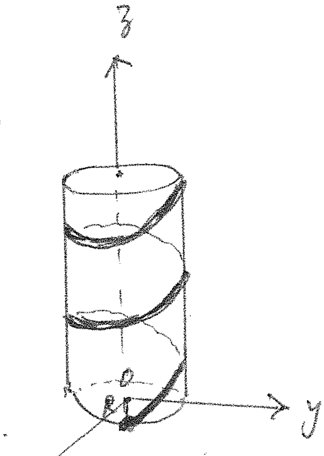
Using vectors, we can define curves/surfaces in $\mathbb{R}^{2,3}$.

First, curves. A curve in $\mathbb{R}^{2,3}$ is the set of position vectors $\vec{v}$ satisfying

(i) $\mathbb{R}^2$ case. $\vec{v} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ where t is some parameter (which can be interpreted as time).

E.g. $\vec{v} = \begin{pmatrix} R\cos t \\ R\sin t \end{pmatrix}$. This describes a circle of radius R centered at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

RMK: We can also write $\vec{v} = (R\cos t, R\sin t)$ or $\vec{v} = R\cos t\,\hat{i} + R\sin t\,\hat{j}$.



(ii) $\mathbb{R}^3$ case. $\vec{v} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$. E.g. $\vec{v} = \begin{pmatrix} R\cos t \\ R\sin t \\ kt \end{pmatrix}$

k is a constant.

RMK: Note that for a curve, only 1 parameter is needed to describe it.

A more mathematical way to describe it is as follows:

$$\begin{array}{ccc} \vec{\alpha}: I & \longrightarrow & \mathbb{R}^{2,3} \\ \Psi & & \Psi \\ t & \longmapsto & \vec{\alpha}(t) \end{array}$$

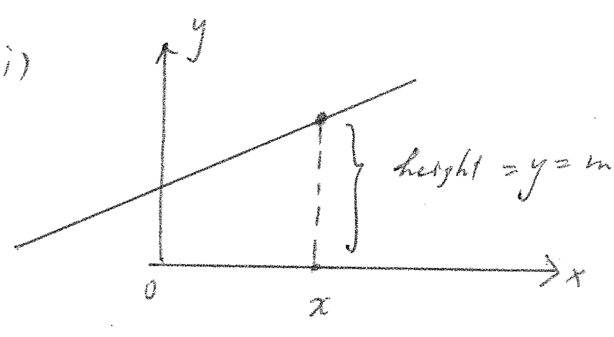
Here I is an interval, e.g. $[a,b]$, (a,b) etc.,
$\vec{\alpha}(t)$ is the position vector of the curve at time t.

(Note that we have used $\vec{v}$ to denote it previously!)

Note also that I is the domain of the "vector-valued" function $\vec{\alpha}$.

E.g.
In school math, we learned that $y = mx + c$ describes a straight line. We can think of this either as (i)

(i)



or (ii)