1(a) Show that

$$n \ll n^2 \ll 2^n \ll 100^n \ll n! \ll n^n$$

where, e.g. $n! \ll n^n$ is to be read as the ratio

$$\lim_n \frac{n!}{n^n} = 0$$

Hints: If $2K \leq n$ then $\frac{n!}{n^n} < \frac{K \cdot (K-1) \cdots 3 \cdot 2 \cdot 1}{n \cdot n \cdots n \cdot n} \leq \left(\frac{1}{2}\right)^K$.

Another useful estimate is

$$(101)^{n-100} \leq n! \quad \text{for all } n \geq 101$$

$\left(\text{so } \frac{100^n}{n!} \leq \frac{100^n}{(101)^{n-100}} \to 0 \text{ as } n \to \infty \right)$.

The ratio test can be used to provide simpler proofs for some problems of similar kind.

(b*). Let $b > 0$. Show that $b^n \ll n!$

2 Show that, as $n \to \infty$,

a*) $n^{1/n} \to 1$ (Hint: write $n^{1/n} = 1 + \delta_n$)

b) $n^{1/n^2} \to 1$ (Hint: Squeeze Th)

c) $(n!)^{1/n^2} \to 1$ $\left(1 < (n!)^{\frac{1}{n^2}} \leq (n^n)^{\frac{1}{n^2}} = n^{1/n} \right)$

3* Let (x_n) be a sequence convergent to $x \in \mathbb{R}$, and let (y_n) be defined by

$$y_n = (x_1 + x_2 + \cdots + x_n)/n, \quad \forall n.$$

Show that $\lim_n y_n = x$ [Hint: Assume $x = 0$ and take a bound M for $(|x_n|)$. Let $\varepsilon > 0$. Then $\exists N' \in \mathbb{N}$ s.t. $|x_n| < \varepsilon/2 \ \forall n \geq N'$. Moreover, $\forall n > N'$, $|y_n| \leq \frac{N'M}{n} + \varepsilon/2$ as

$$y_n = \frac{x_1 + \cdots + x_{N'}}{n} + \frac{x_{N'+1} + \cdots + x_n}{n}$$