Solution to Homework 1

Sec. 1.2

1. (a) Yes. It is condition (VS 3).
   (b) No. If x, y are both zero vectors. Then by condition (VS 3) $x = x + y = y$.
   (c) No. Let e be the zero vector. We have $1e = 2e$.
   (d) No. It will be false when $a = 0$.
   (e) Yes.
   (f) No. It has m rows and n columns.
   (g) No.
   (h) No. For example, we have $x + (-x) = 0$.
   (i) Yes.
   (j) Yes.
   (k) Yes. That is the definition.

8. By (VS 7) and (VS 8), we have

$$(a + b)(x + y) = a(x + y) + b(x + y) = ax + ay + bx + by$$

14. Yes. All the condition are preserved when the field is the real numbers.

15. No. Because a real-valued vector scalar multiply with a complex number will not always be a real-valued vector.

21. Since each entries could be 1 or 0 and there are $m \times n$ entries, there are $2^{m \times n}$ vectors in that space.

Sec. 1.3

1. (a) No. This should make sure that the field and the operations of V and W are the same. Otherwise, for example, $V = \mathbb{R}$ and $w = \mathbb{Q}$ respectively. Then W is a vector space over $\mathbb{Q}$ but not a space over $\mathbb{R}$ So it is not a subspace of V.
   (b) No. We should have that any subspace contains 0.
   (c) Yes. We can choose $W = 0$.
   (d) No. Let $V = \mathbb{R}$, $E_0 = \{0\}$ and $E_1 = \{1\}$. Then we have $E_0 \cap E_1 = \emptyset$ is not a subspace.

(e) Yes. Only entries on diagonal could be nonzero.

(f) No. It is the summation of that.

(g) No. But it is called isomorphism. That is, they are the same in view of structure.

8. Just check whether it's closed under addition and scalar multiplication and whether it contains 0. And here s and t are in $\mathbb{R}$.

   (a) Yes. It is a line $t(3,1,1)$.

   (b) No. It contains no $(0,0,0)$.

   (c) Yes. It is a plane with normal vector $(2,7,1)$.

   (d) Yes. It is a plane with normal vector $(1,4,1)$.

   (e) No. It contains no $(0,0,0)$.

   (f) No. We have both $(\sqrt{3},\sqrt{5},0)$ and $(0,\sqrt{6},\sqrt{3})$ are elements of W_6. But their sum $(\sqrt{3},\sqrt{5}+\sqrt{6},\sqrt{3})$ is not an element of W_6.

11. No in general but yes when $n=1$. Since W is not closed under addition. For example, when $n=2, (x^2+x)+(-x^2)=x$ is not in W.

19. It's easy to say that it is sufficient since if we have $W_1 \subset W_2$ or $W_2 \subset W_1$. Then the union of W_1 and W_2 will be W_1 or W_2, a space of course.

    To say it is necessary we may assume that neither $W_1 \subset W_2$ or $W_2 \subset W_1$ holds. Then we can find some $x \in W_1 - W_2$ and $y \in W_2 - W_1$. Thus by the condition of subspace we have $x+y$ is a vector in W_1 or in W_2, say W_1. But this will make $y=(x+y)-x$ should be in W_1. It will be contradictory to the original hypothesis that $y \in W_2 - W_1$.

23. (a) We have $(x_1+x_2)+(y_1+y_2)=(x_1+y_1)+(x_2+y_2) \in W_1+W_2$ and $c(x_1+x_2)=cx_1+cx_2 \in W_1+W_2$ if $x_1,y_1 \in W_1$ and $x_2,y_2 \in W_2$. And we have $0=0+0 \in W_1+W_2$. Finally $W_1=\{x+0 : x \in W_1, 0 \in W_2\} \subset W_1+W_2$ and it's similar for the case of W_2.

    (b) If U is a subspace contains both W_1 and W_2 then $x+y$ should be a vector in U for all $x \in W_1$ and $y \in W_2$.

28. By the previous exercise we have $(M_1+M_2)^t = M_1^t + M_2^t = -(M_1+M_2)$ and $(cM)^t = cM^t = -cM$. With addition that zero matrix is skew-symmetric we have the set of all skew-symmetric matrices is a space. We have $M_{nn}(\mathbb{F}) = A|A \in M_{nn}(\mathbb{F}) = (A+A^t)+(A-A^t)|A \in M_{nn}(\mathbb{F}) = W_1+W_2$ and $W_1 \cap W_2 = \{0\}$. The final equality is because $A+A^t$ is symmetric and $A-A^t$ is skew-symmetric. If $\mathbb{F}$ is of characteristic 2, we have $W_1=W_2$.

30. If $V=W_1 \oplus W_2$ and some vector $y \in V$ can be represented as $y=x_1+x_2=x_1'+x_2'$, where $x_1,x_1' \in W_1$ and $x_2,x_2' \in W_2$. Then we have $x_1-x_1' \in W_1$ and $x_1-x_1'=x_2'-x_2 \in W_2$. But since $W_1 \cap W_2 = \{0\}$, we have $x_1=x_1'$ and $x_2=x_2'$.

    Conversely, if each vector in V can be uniquely written as x_1+x_2, then $V=W_1+W_2$. Now if $x \in W_1 \cap W_2$ and $x \neq 0$, then we have that $x=x+0$ with $x \in W_1$ and $0 \in W_2$ or $x=0+x$ with $0 \in W_1$ and $x \in W_2$, i.e. a contradiction.

31. (a) If $v + W$ is a space, we have $0 = v + (-v) \in v + W$ and thus $v \in W$ and $v \in W$. Conversely, if $v \in W$ we have actually $v + W = W$, a space.

(b) We can prove that $v_1 + W = v_2 + W$ if and only if $(v_1 - v_2) + W = W$. This is because $(-v_1) + (v_1 + W) = W$ and $(-v_1) + (v_2 + W) = (-v_1 + v_2) + W$. So if $(v_1 - v_2) + W = W$, a space, then we have $v_1 - v_2 \in W$ by the previous exercise. And if $v_1 - v_2 \in W$ we can conclude that $(v_1 - v_2) + W = W$.

(c) We have $(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W = (v_1' + v_2') + W = (v_1' + W) + (v_2' + W)$ since by the previous exercise we have $v_1 - v_1' \in W$ and $v_2 - v_2' \in W$ and thus $(v_1 + v_2) - (v_1' + v_2') \in W$. On the other hand, since $v_1 - v_1' \in W$ implies $av_1 - av_1' = a(v_1 - v_1') \in W$, we have $a(v_1 + W) = a(v_1' + W)$.

(d) It is closed because V is closed. The commutativity and associativity of addition is also because V is commutative and associative. For the zero element we have $(x + W) + W = x + W$. For the inverse element we have $(x + W) + (-x + W) = W$. For the identity element of multiplication we have $1(x + W) = x + W$. The distribution law and combination law are also followed by the original propositions in V. But there are one more thing should be checked, that is whether it is well-defined. But this is the exercise 1.3.31. (c).

Sec. 1.4

1. (a) Yes. Just pick any coefficient to be zero.

(b) No. By definition it should be $\{0\}$.

(c) Yes. Every subspaces of which contains S as a subset contains *span*(S) and *span*(S) is a subspace.

(d) No. This action will change the solution of one system of linear equations.

(e) Yes.

(f) No. For example, $0x = 3$ has no solution.

10. Any symmetric 2×2 matrices can be written as following:

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = aM_1 + cM_2 + bM_3$$

11. For $x \neq 0$ the statement is the definition of linear combination and the set is a line. For $x = 0$ the both side of the equation is the set of zero vector and the set is the origin.

14. We prove $span(S_1 \cup S_2) \subset span(S_1) + span(S_2)$ first. For $v \in span(S_1 \cup S_2)$ we have $v = \sum_{i=1}^{n} a_i x_i + \sum_{j=1}^{m} a_j y_j$ with $x_i \in S_1$ and $y_j \in S_2$. Since the first summation is in $span(S_1)$ and the second summation is in $span(S_2)$, we have $v \in span(S_1) + span(S_2)$.

For the converse, let $u + v \in span(S_1) + span(S_2)$ with $u \in span(S_1)$ and $v \in span(S_2)$. We can write $v = \sum_{i=1}^{n} a_i x_i + \sum_{j=1}^{m} a_j y_j$ with $x_i \in S_1$ and $y_j \in S_2$. This means $u + v \in span(S_1 \cup S_2)$.

Sec. 1.5

1. (a) No. For example, take $S = \{(1,0),(2,0),(0,1)\}$ and then $(0,1)$ is not a linear combination of the other two.
   (b) Yes. It is because $1\vec{0} = \vec{0}$.
   (c) No. It is independent by the remark after Definition of linearly independent.
   (d) No. For example, we have $S = \{(1,0),(2,0),(0,1)\}$ but $\{(1,0),(0,1)\}$ is linearly independent.
   (e) Yes. This is the contrapositive statement of Theorem 1.6.
   (f) Yes. This is the definition.

2. (a) Linearly dependent.
   (b) Linearly independent.
   (c) Linearly independent.
   (d) Linearly dependent.
   (e) Linearly dependent.
   (f) Linearly independent.
   (g) Linearly dependent.
   (h) Linearly independent.
   (i) Linearly independent.
   (j) Linearly dependent.

9. It is sufficient since if $u = tv$ for some $t \in \mathbb{F}$ then we have $utv = 0$. While it is also necessary since if $au + bv = 0$ for some $a, b \in \mathbb{F}$ with at least one of the two coefficients not zero, then we may assume $a \neq 0$ and $u = -\frac{b}{a}v$.

10. Pick $v_1 = (1,1,0), v_2 = (1,0,0), v_3 = (0,1,0)$. And we have that none of the three is a multiple of another and they are dependent since $v_1 - v_2 - v_3 = 0$.

13. (a) Sufficiency: If $\{u+v, u-v\}$ is linearly independent we have $a(u+v)+b(u-v) = 0$ implies $a = b = 0$. Assuming that $cu + dv = 0$, we can deduce that $\frac{c+d}{2}(u+v) + \frac{c-d}{2}(u-v) = 0$ and hence $\frac{c+d}{2} = \frac{c-d}{2} = 0$. This means $c = d = 0$ if the characteristic is not two.

    Necessity: If $\{u, v\}$ is linearly independent we have $au + bv = 0$ implies $a = b = 0$. Assuming that $c(u+v) + d(u-v) = 0$, we can deduce that $(c+d)u + (c-d)v = 0$ and hence $c+d = c-d = 0$. This means $c = d = 0$ if the characteristic is not two.

    (b) Similar to (a).

15. Sufficiency: If $u_1 = 0$, then S is linearly independent. Suppose u_{k+1} is in $\mathrm{span}(\{u_1, u_2, \ldots, u_k\})$ for some k, say $u_{k+1} = a_1u_1 + a_2u_2 + \cdots + a_ku_k$. Then we have $a_1u_1 + a_2u_2 + \cdots + a_ku_k - u_{k+1} = 0$ is a nontrivial representation.

Necessary: If S is linearly dependent, there is some integer k such that there is some nontrivial representation.

$$a_1u_1 + a_2u_2 + \cdots + a_ku_k + a_{k+1}u_{k+1} = 0$$

Furthermore we may assume that $a_{k+1} \neq 0$ otherwise we may choose less k until that $a_{k+1} \neq 0$. Hence we have

$$u_{k+1} = -\frac{1}{a_{k+1}}(a_1u_1 + a_2u_2 + \cdots + a_ku_k)$$

and so $u_{k+1} \in \operatorname{span}(\{u_1, u_2, \ldots, u_k\})$

Sec. 1.6

1. (a) No. The empty set is its basis.
   (b) Yes. This is the result of Replacement Theorem.
   (c) No. For example, the set of all polynomials has no finite basis.
   (d) No. $\mathbb{R}^2$ has $\{(1,0),(1,1)\}$ and $\{(1,0),(0,1)\}$ as bases.
   (e) Yes. This is the Corollary after Replacement Theorem.
   (f) No. It is $n+1$.
   (g) No. It is $m \times n$.
   (h) Yes. This is the Replacement Theorem.
   (i) No. For $S = \{1,2\}$, a subset of $\mathbb{R}$, then $5 = 1\times1+2\times2 = 3\times1+1\times2$.
   (j) Yes. This is Theorem 1.11.
   (k) Yes. It is 0 and V respectively.
   (l) Yes. This is the Corollary 2 after Replacement Theorem.

5. It is impossible since the dimension of $\mathbb{R}^3$ is three.

7. We have first $\{u_1, u_2\}$ is linearly independent. And since $u_3 = -4u_1$ and $u_4 = -3u_1 + 7u_2$, we can check that $\{u_1, u_2, u_5\}$ is linearly independent and hence it is a basis.

11. Since $\{u, v\}$ is a basis for V, $\dim V = 2$. So if $\{u+v, au\}$ is a set of 2 linearly independent vectors, then it is a basis. Now if

$$c_1(u+v) + c_2(au) = 0,$$

then we get

$$(c_1 + ac_2)u + c_1v = 0.$$

By the linear independency of $\{u, v\}$, we have

$$\begin{cases} c_1 + ac_2 & = & 0 \\ c_1 & = & 0 \end{cases}.$$

As a and b are nonzero, we have $c_1 = c_2 = 0$. Hence $\{u+v, au\}$ are linearly independent and is a basis for V.

Similarly $c_1(au) + c_2(bv) = 0$ implies $c_1 = c_2 = 0$. So $\{au, bv\}$ are linearly independent, hence a basis for V.

12. Suppose $\{u, v, w\}$ is a basis for V. Then we know $\dim V = 3$. So $\{u+v+w, v+w, w\}$ is a basis if they are linearly independent. For

$$c_1(u+v+w) + c_2(v+w) + c_3(w) = 0,$$

we can rearrange to get

$$c_1 u + (c_1+c_2)v + (c_1+c_2+c_3)w = 0.$$

By the linear independence of $\{u, v, w\}$, we have $c_1 = c_1+c_2 = c_1+c_2+c_3 = 0$. Then obviously we have $c_1 = c_2 = c_3 = 0$. So $\{u+v+w, v+w, w\}$ are linearly independent and hence a basis.

22. One can see that the dimension of W_1 after intersecting with W_2 does not change. So it is natural to think that W_1 is a set not larger than W_2 and contained in W_2.

    More rigorously, we claim that $W_1 \subset W_2$ is the condition.

    It is obvious to see that $W_1 \subset W_2$ implies $\dim(W_1 \cap W_2) = \dim(W_1)$. Since $W_1 \subset W_2$, $W_1 \cap W_2$ is just W_1. So their dimensions are equal.

    To show the necessary condition, we show that $\dim(W_1 \cap W_2) = \dim(W_1)$ implies $W_1 \subset W_2$. Suppose that W_1 is not contained in W_2, that means there is some vector $v \in W_1 \setminus W_2$. Let β be the basis of $W_1 \cap W_2$, in particular β is in W_1. Note that v is in W_1 but not W_2, so $v \notin \operatorname{span}(\beta)$. But then $\beta \cup \{v\}$ are linearly independent in W_1. So the size of this linearly independent set of vectors is greater than $\dim(W_1)$, which leads to a contradiction. Hence W_1 must be contained in W_2.

23. (a) Note that $W_1 = \operatorname{span}(\{v_1, v_2, \ldots, v_k\})$ has the same dimension as $W_2 = \operatorname{span}(\{v_1, v_2, \ldots, v_k, v\})$. So v cannot help in generating more combinations. In other word v is already in $\operatorname{span}(\{v_1, v_2, \ldots, v_k\})$. Hence we claim that $v \in \{v_1, v_2, \ldots, v_k\} = W_1$ is the condition.

    To show the necessary condition, we suppose $v \notin W_1$. Let β be a basis of W_1. Note that $\beta \subset W_1 \subset W_2$ and $v \in W_2$. Then $\beta \cup \{v\}$ is a linearly independent set in W_2. But then

    $$\dim(W_2) = \dim(W_1) = |\beta| < |\beta| + 1 = |\beta \cup \{v\}| \leq \dim(W_2)$$

    leads to a contradiction. Hence $v \in W_1$.
    To check that the condition is sufficient, if $v \in \{v_1, v_2, \ldots, v_k\}$, then

    $$\operatorname{span}(\{v_1, v_2, \ldots, v_k, v\}) = \operatorname{span}(\{v_1, v_2, \ldots, v_k\}).$$

    Hence W_1 and W_2 are the same, so are their dimensions.

    (b) With the above condition, it shows that $\dim(W_1) \neq \dim(W_2)$ implies $v \notin W_1$. Then following the same arguments in (a), we arrive at

    $$\dim(W_1) = |\beta| < |\beta| + 1 = |\beta \cup \{v\}| \leq \dim(W_2).$$

    Hence $\dim(W_1) < \dim(W_2)$.

26. For any $f \in P_n(\mathbb{R})$ with $f(a) = 0$, we have $f(x) = (x-a)g(x)$ for some $g \in P_{n-1}(\mathbb{R})$. So the subspace

$$\{f \in P_n(\mathbb{R}) : f(a) = 0\}$$

is equivalent to

$$\{f \in P_n(\mathbb{R}) : f(x) = (x-a)g(x), g \in P_{n-1}(\mathbb{R})\}.$$

But $P_{n-1}(\mathbb{R})$ is of dimension n, so is the subspace.

More explicitly, $\{1, x, \ldots, x^{n-1}\}$ is a basis for $P_{n-1}(\mathbb{R})$. One can easily check that $\{(x-a)1, (x-a)x, \ldots, (x-a)x^{n-1}\}$ is a basis for the subspace. Or simply one can observe $\{x-a, x^2-a^2, \ldots, x^n-a^n\}$ is another basis for the subspace.

29. (a) Suppose $\{u_1, u_2, \ldots, u_k\}$ is a basis for $W_1 \cap W_2$. Then one can extend it to a basis $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m\}$ for W_1 and a basis $\{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots, w_p\}$ for W_2. We want to show that $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_p\}$ is a basis for $W_1 + W_2$. Suppose we have

$$\sum_{i=1}^{k} a_i u_i + \sum_{i=1}^{m} b_i v_i + \sum_{i=1}^{p} c_i w_i = 0.$$

One can rearrange to get

$$-\sum_{i=1}^{m} b_i v_i = \sum_{i=1}^{k} a_i u_i + v_i + \sum_{i=1}^{p} c_i w_i.$$

We then see the left hand side is in W_1, while the right hand side is in W_2. So both are in $W_1 \cap W_2$. In particular, $-\sum_{i=1}^{m} b_i$ can be expressed as a combination of $\{u_1, u_2, \ldots, u_k\}$.

$$-\sum_{i=1}^{m} b_i v_i = s_1 u_1 + s_2 u_2 + \cdots + s_k u_k$$

We can rearrange to get

$$s_1 u_1 + s_2 u_2 + \cdots + s_k u_k + b_1 v_1 + b_2 v_2 + \cdots + b_m v_m = 0.$$

Note that $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m\}$ is a basis. The linear independence implies coefficients are all zero. In particular $b_1 = b_2 = \cdots = b_m = 0$. Now the right hand side vanishes

$$0 = \sum_{i=1}^{k} a_i u_i + v_i + \sum_{i=1}^{p} c_i w_i.$$

Again, linear independence of $\{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots, w_p\}$ implies that $a_1 = a_2 = \cdots = a_k = c_1 = c_2 = \cdots = c_p = 0$. Hence

$$\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_p\}$$

are linearly independent.

Now for any vector $z \in W_1 + W_2$, we have $z = x + y$, where $x \in W_1$ and $y \in W_2$. Then we can write x and y as follows.

$$x = a_1 u_1 + a_2 u_2 + \cdots + a_k u_k + b_1 v_1 + b_2 v_2 + \cdots + b_m v_m$$

$$y = c_1 u_1 + c_2 u_2 + \cdots + c_k u_k + d_1 w_1 + d_2 w_2 + \cdots + d_m w_m$$

So we can rearrange the sum to get

$$\begin{aligned} z &= \sum_{i=1}^{k} (a_i + c_i) u_i + \sum_{i=1}^{m} b_i v_i + \sum_{i=1}^{p} d_i w_i \\ &\in \operatorname{span}(\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_p\}) \end{aligned}$$

Hence we have checked $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_p\}$ is a basis for $W_1 + W_2$. Then one can easily check their dimensions.

$$\dim(W_1 + W_2) = k + m + p = \dim(W_1) + \dim(W_2) - \dim(W_1 + W_2)$$

(b) V is the direct sum of W_1 and W_2 if and only if $W_1 \cap W_2 = \{\mathbf{0}\}$. But $W_1 \cap W_2 = \{\mathbf{0}\}$ if and only if $\dim(W_1 \cap W_2) = 0$. From the above formula, we have $\dim(V) = \dim(W_1) + \dim(W_2)$.

Sec. 2.1

15. To show that T is linear, for any $f, g \in P(\mathbb{R})$, we write $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = sum_{i=0}^{m} b_i x^i$. Without loss of generality, we may assume $n >= m$ and set $b_i = 0$ for $i > m$. So we can write $g(x) = sum_{i=0}^{n} b_i x^i$. Then

$$\begin{aligned} T(f(x) + g(x)) = \int_0^x (f(x) + g(x)) dt &= \int_0^x \sum_{i=0}^{n} (a_i + b_i) t^i dt \\ &= \int_0^x \sum_{i=0}^{n} a_i t^i dt + \int_0^x \sum_{i=0}^{n} b_i t^i dt \\ &= T(f(x)) + T(g(x)) \end{aligned}$$

and, similarly, $T(\alpha f(x)) = \alpha T(f(x))$.

To prove that T is one-to-one, we suppose $T(f(x)) = T(g(x))$. Then one can do the integration and get $a_i = b_i$ for all $0 \leq i \leq n$.

$$\int_0^x f(t) dt = \int_0^x g(t) dt$$
$$\sum_{i=0}^{n} \frac{a_i}{i+1} x^{i+1} = \sum_{i=0}^{n} \frac{b_i}{i+1} x^{i+1}$$

Hence $f \equiv g$ and T is one-to-one.

However T is not onto. For example, for any constant $c \in P(\mathbb{R})$ but there is no such $f \in P(\mathbb{R})$ such that $T(f(x)) = \int_0^x f(t) dt = 1$.

17. Let's recall the rank-nullity formula $\dim V = \dim R(T) + \dim N(T)$.

    (a) Suppose $\dim V < \dim W$ but T is onto. Then we have $R(T) = W$. But $\dim N(T) \geq 0$. So

    $$\dim V < \dim W = \dim R(T) <= \dim R(T) + \dim N(T) = \dim V,$$

    which is a contradiction. Hence T cannot be onto.

    (b) Suppose $\dim V > \dim W$ but T is one-to-one. Then we have $N(T) = \{\mathbf{0}\}$ and $\dim N(T) = 0$. But $\dim R(T) \leq \dim W$. So

    $$\dim V > \dim W \geq \dim R(T) = \dim R(T) + \dim N(T) = \dim V,$$

    which is again a contradiction. Hence T cannot be one-to-one.

18. Note that $N(T)$ is in the domain while $R(T)$ is in the codomain. Consider the linear transformation $T(x, y) = (y, 0)$. One can easily check that

    $$N(T) = \{(x, 0) : x \in \mathbb{R}\} = R(T).$$

19. Consider $T(x, y) = (-y, x)$ and $U(x, y) = (2x, 2y)$. Then

    $$R(T) = \{(x, y) \in \mathbb{R}^2\} = R(U)$$

    and

    $$N(T) = \{(0, 0) \in \mathbb{R}^2\} = N(U).$$

    Actually T is a rotation and U is a scaling. One can also try U to be a translation. The conclusion is the same.

Sec. 2.2

13. Suppose $T : V \to W$, $U : V \to W$ and $R(T) \cap R(U) = \{\mathbf{0}\}$. If $\alpha T + \beta U$ is the zero linear transformation from V to W (that is the 0 in $\mathcal{L}(V, W)$), then we want to show that $\alpha = \beta = 0$. That means $(\alpha T + \beta U)(v) = 0$ for all $v \in V$. So

    $$\alpha T(v) = -\beta U(v) \text{ for all } v \in V.$$

    Note that the left hand side is in $R(T)$, while the right hand side is in $R(U)$. Then $R(T) \cap R(U) = \{\mathbf{0}\}$ implies that $\alpha T(v) = 0$ and $-\beta U(v) = 0$ for all v in V. Note that T and U are nonzero linear transformations. Hence it must be that $\alpha = \beta = 0$ and the linear independency follows.

16. Let $n = \dim V$. Suppose $\{v_1, v_2, \ldots, v_k\}$ is a basis for $N(T)$. Then one can extend it to be a basis β for V.

    $$\beta = \{v_1, v_2 \ldots, v_k, v_{k+1}, \ldots, v_n\}$$

    Note that $\{Tv_{k+1}, \ldots, Tv_n\}$ is a basis for $R(T)$. Since $\dim W = \dim V = n$, we can extend the above to be a basis γ for W.

    $$\gamma = \{w_1, w_2, \ldots, w_k, Tv_{k+1}, \ldots, Tv_n\}$$

Now we can check that $[T]_\beta^\gamma$ is diagonal.

$$[T]_\beta^\gamma = \begin{pmatrix} | & | & & | \\ [Tv_1]_\gamma & [Tv_2]_\gamma & \cdots & [Tv_n]_\gamma \\ | & | & & | \end{pmatrix}$$

One can see that $Tv_1, Tv_2, \ldots, Tv_k$ are zero vectors and $Tv_{k+1}, \ldots, Tv_n$ are basis in W. So $\{[Tv_{k+1}]_\gamma, \ldots, [Tv_n]_\gamma\}$ are $\{e_{k+1}, \ldots, e_n\}$, where e_j is a vector with a 1 at jth entry and 0 elsewhere.

$$[T]_\beta^\gamma = \begin{pmatrix} O & O \\ O & I_{n-k} \end{pmatrix}$$

Sec. 2.3

12. Note that UT is a transformation from V to Z.

    (a) Suppose $T(x) = T(y)$ for $x, y \in V$. Then by transforming them with U, we have $UT(x) = UT(y)$. As UT is one-to-one, so $x = y$. Hence T is one-to-one.

    However U may not be one-to-one. One example is that $T(x, y) = (x, y, 0)$ and $U(x, y, z) = (x, y)$. Then T and UT are one-to-one, while U is not.

    (b) Suppose $z \in Z$. As UT is onto, there exists some $v \in V$ such that $UT(v) = z$. Then take $w = T(v)$, we have $U(w) = U(T(v)) = z$. Hence U is onto.

    However T may not be onto. A counter example would be the one in (a).

    (c) For $UT(x) = UT(y)$, as U is one-to-one, we have $T(x) = T(y)$. Again as T is one-to-one, we have $x = y$. So UT is one-to-one. For $z \in Z$, as U is onto, there exists some $w \in W$ such that $U(w) = z$. Then for this w, as T is onto, there exist some $v \in V$ such that $T(v) = w$. Hence we have found $v \in V$ such that $UT(v) = U(w) = z$. So UT is onto.

13. Denote $(M)_{ij}$ as the entry at ith row and jth column. Then one can simply write from definition to check that they are equal.

$$\begin{aligned} \operatorname{tr}(AB) &= \sum_{i=1}^n (AB)_{ii} = \sum_{i=1}^n \sum_{j=1}^n (A)_{ij}(B)_{ji} \\ \operatorname{tr}(BA) &= \sum_{j=1}^n (BA)_{jj} = \sum_{j=1}^n \sum_{i=1}^n (A)_{ji}(B)_{ij} = \sum_{i=1}^n \sum_{j=1}^n (A)_{ij}(B)_{ji} \end{aligned}$$

    As the sum is finite, we can interchange the order of summations and get the result.

17. Suppose $T : V \to V$ is a linear transformation such that $T = T^2$. Denote

$$W_1 = \{y \in V : T(y) = y\} \text{ and } W_2 = \{y \in V : T(y) = 0\}(= N(T)).$$

Note that for every vector $x \in V$, we can write $x = T(x) + (x - T(x))$. In other words, we write

$$x = u + v, \text{ where } u = T(x) \text{ and } v = x - T(x).$$

We claim that $u \in W_1$ and $v \in W_2$. Firstly, $T(u) = T(T(x)) = T(x) = u$, so $u \in W_1$. Next, $T(v) = T(x - T(x)) = T(x) - T(T(x)) = T(x) - T(x) = 0$, so $v \in W_2$.

Moreover we can see that

$$V = W_1 \oplus W_2,$$

that means $W_1 \cap W_2 = \{\mathbf{0}\}$. For any $y \in W_1 \cap W_2$, we have both $T(y) = y$ and $T(y) = 0$, so $y = 0$.

To interpret this, V is decomposed into two subspaces, say $V = W_1 \oplus W_2$. A vector $v \in V$ can be expressed as a sum of two parts, one from each subspace. A transformation T will preserve the first part and remove the other part, that is just a projection. Hence T is a projection of vectors onto some subspace $W1 \subset V$ which $V = W_1 \oplus W_2$.

Sec. 2.4

4. As A and B are invertible, there exists A^{-1} and B^{-1} such that

$$A^{-1}A = AA^{-1} = I \text{ and } B^{-1}B = BB^{-1} = I.$$

So it is easy to check

$$B^{-1}A^{-1}AB = I \text{ and } ABB^{-1}A^{-1} = I.$$

Hence AB is invertible and $B^{-1}A^{-1}$ is the inverse of AB. In other words $(AB)^{-1} = B^{-1}A^{-1}$.

16. Φ is an isomorphism if it is a one-to-one onto linear transformation. It is easy to check that Φ is linear.

$$\Phi(C + D) = B^{-1}(C + D)B = B^{-1}CB + B^{-1}DB = \Phi(C) + \Phi(D)$$

$$\Phi(\alpha C) = B^{-1}(\alpha C)B = \alpha B^{-1}CB = \alpha\Phi(C)$$

Note that B is invertible. Then for any C and D such that $\Phi(C) = \Phi(D)$, we have $B^{-1}CB = B^{-1}DB$. Hence $C = D$ and Φ is one-to-one.

Moreover, given A, we can also find $C = BAB^{-1}$ such that $\Phi(C) = B^{-1}BAB^{-1}B = A$. Hence Φ is onto. So it is an isomorphism.

Sec. 3.2

2. (g) By elementary row operations, we get the RREF (reduced row echelon form).

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So the rank is just 1.

5. (h) By computing the RREF of $(A|I)$, we see the rank is just 3. So the inverse does not exist.

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Sec. 3.3

3. (g) Consider

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 1 \\ x_2 - x_3 + x_4 = 1 \end{cases}$$

One particular solution is $x_1 = x_2 = x_3 = 0$ and $x_4 = 1$. Now we consider the homogenous system

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ x_2 - x_3 + x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -3x_3 + x_4 \\ x_2 = x_3 - x_4 \end{cases}$$

We choose $(-3, 1, 1, 0)$ and $(1, -1, 0, 1)$ to be a basis for the solution set of the homogenous system. Then all the solutions can be written as $(0, 0, 0, 1) + s(-3, 1, 1, 0) + t(1, -1, 0, 1)$. Hence the solutions to the system are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3s + t \\ s - t \\ s \\ t + 1 \end{pmatrix} \quad \text{for all } s, t \in \mathbb{R}.$$

4. (b) First, rewrite the system of linear equations as $Ax = b$.

$$\begin{cases} x_1 + 2x_2 - x_3 = 5 \\ x_1 + x_2 + x_3 = 1 \\ 2x_1 - 2x_2 + x_3 = 4 \end{cases} \Rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}$$

i. By computing the RREF of $(A|I)$,

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{9} & \frac{1}{3} & -\frac{2}{9} \\ 0 & 0 & 1 & -\frac{4}{9} & \frac{2}{3} & -\frac{1}{9} \end{array}\right),$$

we get

$$A^{-1} = \frac{1}{9}\begin{pmatrix} 3 & 0 & 3 \\ 1 & 3 & -2 \\ -4 & 6 & -1 \end{pmatrix}.$$

ii. As we now have A^{-1}, x is just $A^{-1}b$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 3 & 0 & 3 \\ 1 & 3 & -2 \\ -4 & 6 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$$

Sec. 3.4

12. (a) Check that $(0, -1, 0, 1, 1, 0)$ and $(1, 0, 1, 1, 1, 0)$ are solutions to the system, so $S \subset V$. Suppose

$$a(0, -1, 0, 1, 1, 0) + b(1, 0, 1, 1, 1, 0) = (0, 0, 0, 0, 0, 0).$$

One can easily solve that $a = b = 0$. Hence they are linearly independent.

(b) Consider the system

$$\begin{cases} x_1 & -x_2 & & +2x_4 & -3x_5 & +x_6 & = 0 \\ 2x_1 & -x_2 & -x_3 & +3x_4 & -4x_5 & +4x_6 & = 0 \end{cases}.$$

One can solve for any two of the variables, say x_1 and x_2.

$$\begin{cases} x_1 = & x_3 & -x_4 & +x_5 & -3x_6 \\ x_2 = & x_3 & +x_4 & -2x_5 & -2x_6 \end{cases}$$

We then get a set of basis for the solution set by setting each of $\{x_3, x_4, x_5, x_6\}$ to be 1.

$$\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

To extend S to a basis for V, we compute the RREF to select two more linearly independent vectors to form a basis.

$$\begin{pmatrix} 0 & 1 & 1 & -1 & 1 & -3 \\ -1 & 0 & 1 & 1 & -2 & -2 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} \mathbf{1} & 0 & -1 & 0 & 1 & 0 \\ 0 & \mathbf{1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence we choose $(-1, 1, 0, 1, 0, 0)$ and $(-3, -2, 0, 0, 0, 1)$. Therefore, the extended basis is

$$\left\{ \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Sec. 4.1

For Exercise 5, 7 and 8, we introduce the following notations.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

5. Suppose B is obtained by interchanging the rows of A.

$$B = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

Then the determinants can be easily computed.

$$\det(B) = cb - ad = -(ad - bc) = -\det(A)$$

7. Note that

$$A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Then the determinants are equal.

$$\det(A^t) = ad - bc = \det(A)$$

8. As A is upper triangular, we have $c = 0$. Then

$$\det(A) = ad - bc = ad,$$

which is the product of the diagonal entries of A.

Sec. 4.2

26. By negating a row of the matrix A, we change $\det(A)$ by a factor of -1. To obtain $-A$, we can negate every row of A, that is negating n times. So

$$\det(-A) = (-1)^n \det(A).$$

For $\det(-A) = \det(A)$, either $\det(A) = 0$, which means A is not invertible; or $(-1)^n = 1$, which means n is even or $\mathbb{F}$ is of characteristic 2.

27. It is obvious that A is not of full-rank, so $\det(A) = 0$.

Sec. 4.3

12. Note that $\det(Q^t) = \det(Q)$. So if $QQ^t = I$, we have $\det(QQ^t) = \det(I)$. Then
$$\det(Q)^2 = \det(Q)\det(Q^t) = \det(QQ^t) = \det(I) = 1.$$
Hence $\det(Q) = \pm 1$.

15. Note that $\det(Q^{-1}) = \det(Q)^{-1}$ for invertible matrix $Q \in M_{n\times n}(\mathbb{F})$. If A and B are similar, there exists some invertible matrix Q such that $A = Q^{-1}BQ$. So
$$\det(A) = \det(Q^{-1}BQ) = \det(Q^{-1})\det(B)\det(Q) = \det(B).$$

22. (a) Note that $\beta = \{1, x, \ldots, x^n\}$ and $\gamma = \{e_1, e_2, \ldots, e_{n+1}\}$.
$$[T]_\beta^\gamma = \begin{pmatrix} | & | & | & & | \\ [T(1)]_\gamma & [T(x)]_\gamma & [T(x^2)]_\gamma & \cdots & [T(x^n)]_\gamma \\ | & | & | & & | \end{pmatrix}$$
For $T(x^j) = (c_0^j, c_1^j, \ldots, c_n^j)$ for $j = 0, 1, 2, \ldots, n$. As γ is just the standard basis, $T(x^j)$ as its usual representation under γ. Hence we have
$$M = [T]_\beta^\gamma = \begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & c_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix}.$$
(b) Using Exercise 22 from Sec. 2.4, we know that T is an isomorphism. In other words, T is an invertible transformation, so is M. Hence $\det(M) \neq 0$.
(c) We recall that
$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}).$$

Then we are ready to expand the determinant.

$$\begin{aligned}
\det(M) &= \det \begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & c_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix} \\
&= \det \begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 0 & c_1 - c_0 & c_1^2 - c_0^2 & \cdots & c_1^n - c_0^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & c_n - c_0 & c_n^2 - c_0^2 & \cdots & c_n^n - c_0^n \end{pmatrix} \\
&= \det \begin{pmatrix} c_1 - c_0 & c_1^2 - c_0^2 & \cdots & c_1^n - c_0^n \\ \vdots & \vdots & \ddots & \vdots \\ c_n - c_0 & c_n^2 - c_0^2 & \cdots & c_n^n - c_0^n \end{pmatrix} \\
&= \prod_{i=1}^{n} (c_i - c_0) \det \begin{pmatrix} 1 & c_1 + c_0 & \cdots & \sum_{i=0}^{n-1} c_1^{n-1-i} c_0^i \\ \vdots & \vdots & \ddots & \vdots \\ 1 & c_n + c_0 & \cdots & \sum_{i=0}^{n-1} c_n^{n-1-i} c_0^i \end{pmatrix}
\end{aligned}$$

Observe that the second column can be reduced by a multiple of the first column.

$$\det(M) = \prod_{i=1}^{n} (c_i - c_0) \det \begin{pmatrix} 1 & c_1 & c_1^2 + c_1 c_0 + c_0^2 & \cdots & \sum_{i=0}^{n-1} c_1^{n-1-i} c_0^i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c_n & c_n^2 + c_n c_0 + c_0^2 & \cdots & \sum_{i=0}^{n-1} c_n^{n-1-i} c_0^i \end{pmatrix}$$

In the same way, we can reduce the jth column by multiples of previous columns. By these column operations, we can get

$$\det(M) = \prod_{i=1}^{n} (c_i - c_0) \det \begin{pmatrix} 1 & c_1 & c_1^2 & \cdots & c_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^{n-1} \end{pmatrix}$$

In the same fashion, we expand this determinant.

$$\begin{aligned}
\det(M) &= \prod_{i=1}^{n} (c_i - c_0) \prod_{i=2}^{n} (c_i - c_1) \det \begin{pmatrix} 1 & c_2 & \cdots & c_2^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & c_n & \cdots & c_n^{n-2} \end{pmatrix} \\
&\vdots \\
&= \prod_{i=1}^{n} (c_i - c_0) \prod_{i=2}^{n} (c_i - c_1) \cdots \prod_{i=n}^{n} (c_i - c_{n-1}) \\
&= \prod_{0 \le i, j \le n} (c_j - c_i)
\end{aligned}$$

Sec. 4.4

1. Basically these are the major "rules" for us to handle determinants. Please make sure you don't get them wrong.

(a) True.
(b) False. (It depends.)
(c) True.
(d) False.
(e) False.
(f) True.
(g) True.
(h) False.
(i) True.
(j) True.
(k) True.

And, of course, please make sure you understand the reasons behind each fact.