Lecture 7: Eigenspace

Corollary 1: Let $T: V \to V$, $\dim(V) = n$. If T has n distinct eigenvalues, then T is diagonalizable.

Proof: Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ are n distinct eigenvalues.

The corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent.

Thus, $\beta := \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ forms a basis of V, which are eigenvectors.

$$\therefore [T]_\beta = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \\ & & & \lambda_n \end{pmatrix}. \quad (T \text{ is diagonalizable})$$

Example 1: Consider again $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Char. poly $= f(t) = (t-1)(t-3)$. $\therefore$ Eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = 3$ (distinct)

$\therefore A$ is diagonalizable.

In fact, $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ = eigenvector of $\lambda_1 = 1$.

$\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ = eigenvector of $\lambda_2 = 3$.

Then: $D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = Q^{-1} A \underbrace{Q}_{[\vec{v}_1, \vec{v}_2]} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

Remark:
- n distinct eigenvalues + $\dim(V) = n \Rightarrow$ Diagonalizable.
- Diagonalizable + $\dim(V) = n \not\Rightarrow n$ distinct eigenvalues.

Ex: $I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ = diagonalizable BUT eigenvalues $= 1$ ONLY.

<u>Definition</u>: A polynomial $f(t)$ splits over $\mathbb{F}$ if:

$$f(t) = c(t-a_1)(t-a_2)\cdots(t-a_n) \quad (a_i \in \mathbb{F})$$

<u>Note</u>: $a_1, a_2, \ldots, a_n$ NEED NOT to be distinct.

e.g. $(t^2+4)(t+1)$ does not split in $\mathbb{R}$.

<u>Theorem 1</u>: The characteristic polynomial of a diagonalizable operator splits.

<u>Proof</u>: Let $T: V \to V$, diagonalizable $\Rightarrow \exists$ ordered basis $\beta \ni$

$$[T]_\beta = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \therefore f(t) = \det([T]_\beta - \lambda I) = (\lambda_1 - t)\cdots(\lambda_n - t)$$

$$= (-1)^n (t-\lambda_1)\cdots(t-\lambda_n).$$

<u>Remark</u>:
- T diagonalizable but DOES NOT have n distinct eigenvalues $\Rightarrow$ Repeated zeros in $f(t)$.
- Diagonalizable $\Rightarrow$ $f(t)$ splits.
  $\not\Leftarrow$ (Example later)

<u>Definition 3</u>: Let λ = eigenvalue of T w/ char poly $f(t)$. The multiplicity of λ = largest integer k for which $(t-\lambda)^k$ is a factor of $f(t)$.

<u>Example 2</u>: $A = \begin{pmatrix} 5 & 1 & & & 0 \\ & 4 & 2 & & \\ & & 3 & 3 & \\ & & & 3 & 4 \\ 0 & & & & 2 \end{pmatrix}$. Char poly: $f(t) = (5-t)(4-t)(3-t)^2(2-t)$

Then: eigenvalues: $\lambda_1 = 5$, $\lambda_2 = 4$, $\lambda_3 = 3$, $\lambda_4 = 2$

Multiplicity: 1 1 2 1

<u>Observation</u>: T diagonalizable w.r.t. $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$

Let $[T]_\beta = \begin{pmatrix} \lambda_1 & & & & & & \\ & \lambda_2 & & & & & \\ & & \ddots & & & & \\ & & & \lambda_k & & & \\ & & & & \lambda_k & & \\ & & & & & \ddots & \\ & & & & & & \lambda_k & \\ & & & & & & & \ddots \\ & & & & & & & & \lambda_n \end{pmatrix}$ with the λ_k in positions $\underbrace{i, \; i+1, \; \ldots, \; i+m-1}_{m \ \lambda_k}$

Then: $\vec{v}_i, \vec{v}_{i+1}, \ldots, \vec{v}_{i+m-1}$ are eigenvector of λ_k and are lin. independent.

Also, all $\vec{v}_i, \vec{v}_{i+1}, \ldots, \vec{v}_{i+m-1} \in N(T - \lambda_k I)$

$\Rightarrow \exists\ m$ lin. independent elements in $N(T - \lambda_k I)$

$\therefore$ $\boxed{\text{Understanding } \dim(N(T - \lambda_k I)) \text{ is important.}}$

<u>Definition 4</u>: Let $T: V \to V$. Define:

$$E_\lambda = \{\vec{x} \in V : T(\vec{x}) = \lambda \vec{x}\} = N(T - \lambda I)$$

<u>Then</u>: E_λ = eigenspace of T.

Similarly, eigenspace of $A \in M_{n \times n}(\mathbb{F})$ = eigenspace of L_A

$= N(A - \lambda I)$

<u>Theorem 3</u>: Let $T: V \to V$ ← finite dim. Let λ = eigenvalue of T w/ multiplicity m. Then: $1 \leq \dim(E_\lambda) \leq m$

<u>Proof</u>: Let $\{\vec{v}_1, \ldots, \vec{v}_p\}$ = ordered bases of E_λ. $\therefore \dim(E_\lambda) = p$.

Extend it to the bases of V: $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p, \vec{v}_{p+1}, \ldots, \vec{v}_n\}$.

Then: $A := [T]_\beta = \left(\begin{array}{ccc|c} \lambda & & & \\ & \ddots & & B \\ & & \lambda & \\ \hline & O & & C \end{array}\right)$

Char poly: $f(t) = \det(A - tI) = \left(\begin{array}{ccc|c} \lambda-t & & & \\ & \ddots & & B \\ & & \lambda-t & \\ \hline & O & & (C - tI_{n-p}) \end{array}\right)$

$$= (\lambda-t)^p \underbrace{\det(C - tI_{n-p})}_{g(t)}$$

So, $(\lambda-t)^p$ is a factor of $f(t)$. $\therefore$ $p \leq m$

$\therefore$ $1 \leq \dim(E_\lambda) \leq m$. Note that $\exists \vec{v} \neq 0 \ni \vec{v} \in E_\lambda \therefore \dim(E_\lambda) \geq 1$.

Example 3: Let $T: P_n(\mathbb{R}) \to P_n(\mathbb{R})$. Define: $T(f(x)) = f'(x)$

Let $\beta = \{1, x, x^2, \ldots, x^n\}$. Then:

$T(1) = 0$, $T(x) = 1$, $T(x^2) = 2x$, $\ldots$, $T(x^n) = nx^{n-1}$.

$\therefore [T]_\beta = \begin{pmatrix} 0 & 1 & & & O \\ & \ddots & 2 & & \\ & & \ddots & \ddots & \\ & O & & \ddots & n-1 \\ & & & & 0 \end{pmatrix}$. Clearly, char poly $= \det([T]_\beta - tI_n)$

$$= (-1)^{n+1} t^{n+1}.$$

$\therefore$ T has only one eigenvalue w/ multiplicity $n+1$.

Now, $E_\lambda = \{f \in P_n(\mathbb{R}) \ni f'(x) = \lambda (= 0)\} = \text{span}\{1\}$.

$\therefore$ $\{1\}$ is the basis of E_λ. $\therefore$ $\dim(E_\lambda) = 1$.

Recall that T is diagonalizable $\Leftrightarrow$ $\exists$ basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ which are eigenvectors

But there cannot be basis of $P_n(\mathbb{R})$ which are eigenvectors (of eigenvalue $= 0$). $\therefore$ T is NOT diagonalizable.

Remark: Hence, char poly splits $\not\Rightarrow$ T is diagonalizable.

Example 2: Consider: $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ such that

$$T(a+bx+cx^2) = (4a+c) + (2a+3b+2c)x + (a+4c)x^2$$

Let $\beta = \{1, x, x^2\}$ = ordered basis of T.

Let $A := [T]_\beta = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}$. Char poly $= f(t) = (5-t)(3-t)^2$

$\therefore$ Eigenvalues: $\lambda_1 = 5$ (Mult = 1)
$\lambda_2 = 3$ (Mult = 2)

Consider $\lambda_1 = 5$, $N(A-5I) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$
$= \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$

$\therefore E_5 = N(T-5I) = \operatorname{span}\{(1+2x+x^2)\}$ and $\dim(E_5) = 1$.

For $\lambda_2 = 3$, $N(A-3I) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\therefore E_3 = N(T-3I) = \operatorname{span}\{x, -1+x^2\}$. $\therefore \dim(E_3) = 2$.

Observe that $\beta_2 = \{1+2x+x^2, x, -1+x^2\}$ forms a basis of $P_2(\mathbb{R})$.

$\therefore$ T is diagonalizable using β_2.