Lecture 21: Unitarily / orthogonally equivalent

Definition: Let $A, B \in M_{n\times n}(\mathbb{C})$ ($M_{n\times n}(\mathbb{R})$)
We say that A and B are unitarily (orthogonally) equivalent if and only if there exists a unitary (orthogonal) matrix P such that $A = P^*BP$ ($A = P^TBP$)

Theorem 1: Let A be a $n\times n$ complex matrix. Then: A is normal if and only if A is unitarily equivalent to a diagonal matrix.

Proof: ($\Rightarrow$) has already been shown from the observation.

($\Leftarrow$) Suppose $A = P^*DP$ where P is unitary and D is diagonal.

Need to show: $A^*A = AA^*$ (normal)

Now,
$$AA^* = (P^*DP)(P^*DP)^* = P^*D\underbrace{PP^*}_{I}D^*P$$
$$= P^*DD^*P$$
$$= P^*D^*DP = A^*A.$$

$\therefore\ AA^* = A^*A$.

Similarly, we have:

Theorem 2: Let A be a $n\times n$ real matrix. Then: A is symmetric (self-adjoint) if and only if A is orthogonally equivalent to a diagonal matrix.

Proof: Exercise.

Another version of Schur's Lemma

Let $A \in M_{n \times n}(\mathbb{F})$. Suppose the char poly splits. Then:

(a) If $\mathbb{F} = \mathbb{C}$, then A is unitarily equivalent to a complex upper triangular matrix.

(b) If $\mathbb{F} = \mathbb{R}$, then A is orthogonally equivalent to a real upper triangular matrix.