The Chinese University of Hong Kong
Department of Mathematics

MMAT 5140 Probability Theory 2015 - 2016
Suggested Solution to Homework 6

1. P. 258, Q1 Note that

$$P(X \le t) = \begin{cases} t & \text{if } 0 < t \le 1, \\ 0 & \text{if } t \le 0, \\ 1 & \text{if } t > 1. \end{cases}$$

Note that $Y > 0$ so it suffices to consider $Y \le t$ for $t > 0$. For $t > 0$

$$\begin{aligned} P(Y \le t) &= P\left(X \ge \frac{1}{t}\right) \\ &= 1 - P\left(X < \frac{1}{t}\right) \\ &= 1 - P\left(X \le \frac{1}{t}\right) \quad (*) \\ &= \begin{cases} 0 & \text{if } 0 < t \le 1. \\ 1 - \frac{1}{t} & \text{if } t > 1. \end{cases} \end{aligned}$$

Hence,

$$f_Y(t) = \begin{cases} 0 & \text{if } t \le 1, \\ \frac{1}{t^2} & \text{if } t > 1. \end{cases}$$

(*) Note that

$$P(X < t) = P\left(\bigcup_{q < t, q \in \mathbb{Q}} \{X \le q\}\right) = P(X \le t)$$

since $P(X \le t)$ is continuous in t.

2. P. 259, Q7

$$\begin{aligned} P(X \le x) &= \begin{cases} 0 & \text{if } x \le 0, \\ \int_0^x 30t^2(1-t)^2 dt & \text{if } 0 < x \le 1, \\ 1 & \text{if } x > 1, \end{cases} \\ &= \begin{cases} 0 & \text{if } x \le 0, \\ 10x^3 - 15x^4 + 6x^5 & \text{if } 0 < x \le 1, \\ 1 & \text{if } x > 1, \end{cases} \end{aligned}$$

Note that $Y \geq 0$ so it suffices to consider $Y \leq y$ for $y \geq 0$.

$$\begin{aligned}
P(Y \leq y) &= P(X^4 \leq y) \\
&= P(-y^{\frac{1}{4}} \leq X \leq y^{\frac{1}{4}}) \\
&= P(X \leq y^{\frac{1}{4}}) \\
&= \begin{cases} 0 & \text{if } y \leq 0, \\ 10y^{\frac{3}{4}} - 15y + 6y^{\frac{5}{4}} & \text{if } 0 < y \leq 1, \\ 1 & \text{if } y > 1, \end{cases} \\
f_Y(y) &= \begin{cases} \frac{15}{2}y^{-\frac{1}{4}} - 15 + \frac{15}{2}y^{\frac{1}{4}} & \text{if } 0 < y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} \frac{15}{2}y^{-\frac{1}{4}}(1 - y^{\frac{1}{4}})^2 & \text{if } 0 < y \leq 1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Note that you can define whatever value for f_Y at 0 because no matter what value one assign to this point, the distribution will never change since a single point has probability 0.

3. P. 281, Q3 For $x \geq 0$,

$$\begin{aligned}
P(|Z| \leq x) &= P(-x \leq Z \leq x) \\
&= P(Z \leq x) - P(Z < -x) \\
&= P(Z \leq x) - (1 - P(Z \geq -x)) \\
&= P(Z \leq x) + P(Z \geq -x) - 1 \\
&= 2P(Z \leq x) - 1 \\
&= 2\phi(x) - 1.
\end{aligned}$$

4. P. 281, Q7 Let $X \sim \mathcal{N}(67, 8^2)$.

   (a) $P(X \geq 90) = P\left(\mathcal{N}(0,1) \geq \frac{90-67}{8}\right) = P(\mathcal{N}(0,1) \geq 2.875) = 1 - \Phi(2.875)$.

   (b) $P(80 \leq X < 90) = P(1.625 \leq \mathcal{N}(0,1) < 2.875) = \Phi(2.875) - \Phi(1.625)$.

   (c) $P(70 \leq X < 80) = P(0.375 \leq \mathcal{N}(0,1) < 1.625) = \Phi(1.625) - \Phi(0.375)$.

   (d) $P(60 \leq X < 70) = P(-0.875 \leq \mathcal{N}(0,1) < 0.375) = \Phi(0.375) - \Phi(-0.875)$.

   (e) $P(X < 60) = P(\mathcal{N}(0,1) < -0.875) = \Phi(-0.875)$.