TUT 7

Def: $X = X_0$ is an isolated critical point of a system if $\exists$ some circle contains only 1 critical point (this is X_0).

Def: The system $\vec{X}' = A\vec{X} + \vec{g}(x)$ (corresponding linear part) A is constant matrix is almost linear at critical point $X = 0$ if $\frac{\|g(x)\|}{\|x\|} \to 0$ as $X \to 0$. (translate the system if $X = X_0 \neq 0$). $|A| \neq 0$.

We consider a general case,

$$\frac{dx}{dt} = F(x,y) \quad \frac{dy}{dt} = G(x,y) \quad -①$$

critical point

Thm: This system is almost linear at $\vec{X} = (x_0, y_0)$ if F and G have cts partial derivatives up to order 2.

Sketch of Proof: By taylor expansion

$$F(x,y) = F(x_0,y_0) + F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0) + \underbrace{\frac{1}{2!}\left((x-x_0)^2 F_{xx}(x_0',y_0') + 2(x-x_0)(y-y_0)F_{xy}(x_0',y_0') + (y-y_0)^2 F_{yy}(x_0',y_0')\right)}_{\eta_F}$$

for some (x_0', y_0')

Similarly for $G(x,y)$.

So the system ① reduced to (Corresponding linear point)

$$\frac{d}{dt}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} = \begin{pmatrix} F_x(\vec{x}_0) & F_y(\vec{x}_0) \\ G_x(\vec{x}_0) & G_y(\vec{x}_0) \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} + \begin{pmatrix} \eta_F \\ \eta_G \end{pmatrix}.$$

Since η_F and η_G contain $(x-x_0)^2, (y-y_0)^2$,

so $\dfrac{\|\eta\|}{\|\vec{x}-\vec{x}_0\|} \to 0$ as $\vec{x} \to \vec{x}_0$.

evalues r_1, r_2	Linear system Type	Linear system Stability	Almost linear system Type	Almost linear system Stability
$r = \pm i\mu$	Centre	stable	Centre or spiral point	indeterminate.
other case	...	....	same as Linear (almost).	same as Linear

Liapunov's second method,

Consider the system ①,

Suppose $\vec{x}=0$ is a critical point.

Def: Let $V: \mathbb{R}^2 \to \mathbb{R}$,

V is +ve definite on $D \subseteq \mathbb{R}^2$ if $V(0,0)=0$

and $V(x,y) > 0 \ \forall (x,y) \in D \setminus \{\vec{0}\}$.

V is -ve definite similarly.

Suppose $x = \phi(t)$, $y = \psi(t)$ is a solution,

$$\dot{V}(x,y) = \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial V}{\partial y} \cdot \frac{dy}{dt}$$
$$= V_x F + V_y G.$$

Thm : Suppose ① has an isolated critical point $\vec{x} = \vec{0}$, and $\exists\, V : \mathbb{R}^2 \to \mathbb{R}$ is cts and has cts 1st partial derivates, V is +ve definite, $\dot{V}$ is -ve definite ($\dot{V} < 0$) on D containing 0, then $x = 0$ is asy. stable.

If $\dot{V}$ is -ve semidefinite, $x = 0$ is stable.

Most of the time, we assume $V = ax^2 + 2bxy + y^2$.

Example : Show $\vec{0}$ is asy stable to the system $\frac{dx}{dt} = y$ $\frac{dy}{dt} = -y - \sin x$

pf: 0 is an isolated critical point

then try $V(x,y) = ax^2 + 2bxy + y^2$

we need to find a, b s.t. ① $V(x,y)$: +ve definite
② $\dot{V}(x,y) < 0$
in some D contain $\vec{0}$.

① : $\begin{cases} a>0 \\ 4a-4b^2>0 \end{cases}$ $\left(V = ax^2 + bxy + cy^2 \text{ is +ve definite iff } \begin{cases} a>0 \\ 4ac-b^2>0. \end{cases} \right)$

$\Rightarrow \begin{cases} a>0 \\ a>b^2 \end{cases}$

$\Rightarrow a > b^2.$

② $\dot{V}(x,y) = (2ax+2by)y - (y+\sin x)(2bx+2y)$

$= 2(axy + by^2 - bxy - y^2 - bx\sin x - y\sin x).$

by taylor series of $\sin x = x - \frac{x^3}{6}\cos t$

$= 2\left(xy(a-b-1) + y^2(b-1) - bx^2 + \frac{bx^4}{6}\cos t + \frac{yx^3}{6}\cos t \right)$

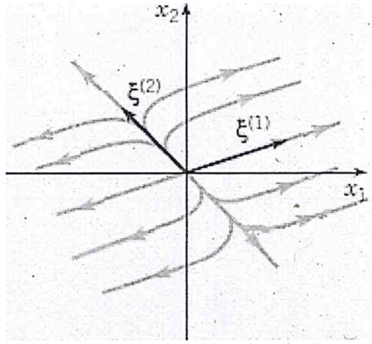
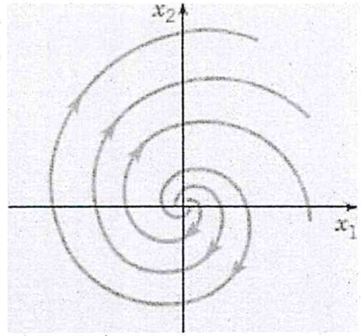
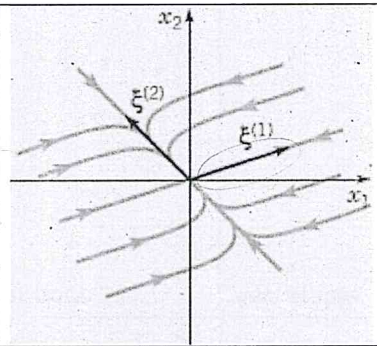
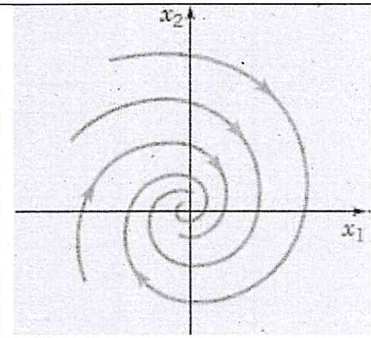
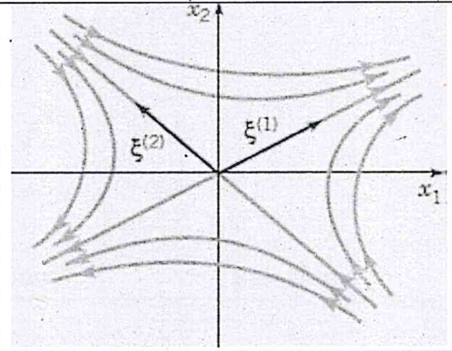
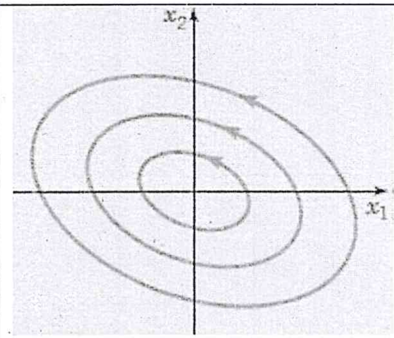
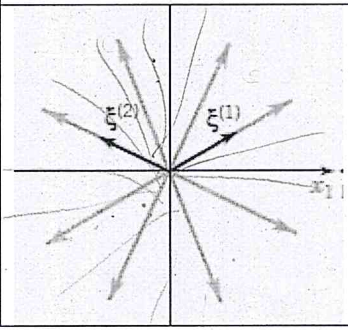
$\leq 2\left(xy(a-b-1) + y^2(b-1) - bx^2 + \frac{bx^2}{6} + \frac{x^2}{6} \right)$

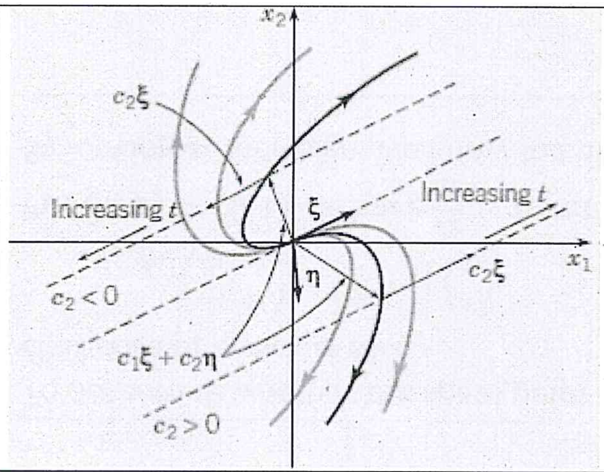
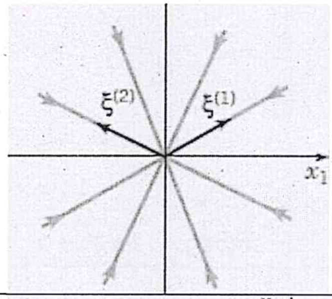
(we assume D is a small region s.t. $|x|, |y| < 1$, and $\cos t < 1$)

Choose $a-b-1=0$, $b=\frac{1}{2}$, then $a=\frac{3}{2}$

$= 2\left(-\frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{x^2}{12} + \frac{x^2}{6} \right) < 0.$ $\forall (x,y) \neq \vec{0}$ and $(x,y) \in D$.

Problem ①: Show $\vec{0}$ is asy stable to $x'' + (1+x^2)x' + \sin x = 0$ by using $V = Ax^2 + xy + y^2$. (let $y = x'$).

$r_1 > r_2 > 0$	Node	Unstable		$r = \lambda \pm i\mu$ $\lambda > 0$	Spiral point	Unstable	
$r_1 < r_2 < 0$	Node	Asy. stable		$r = \lambda \pm i\mu$ $\lambda < 0$	Spiral point	Asy. stable	
$r_2 < 0 < r_1$	Saddle point	Unstable		$r = \pm i\mu$	Center	Stable	
$r_1 = r_2 > 0$ (2 linearly independen t eigenvector s)	Proper node	Unstable		To determine whether the spiral point and the center is clockwise or anticlockwise. $$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$ Put $(x, y) = (0,1)$ we have $\frac{dx}{dt} = b$ and $\frac{dy}{dt} = d$ By looking at their sign ,we know the direction.			

$r_1 = r_2 > 0$ (only 1 linearly independent eigenvector)	Improper node	Unstable	
$r_1 = r_2 < 0$ (2 linearly independent eigenvectors)	Proper node	Asy. stable	
$r_1 = r_2 < 0$ (only 1 linearly independent eigenvector)	Improper node	Asy. stable