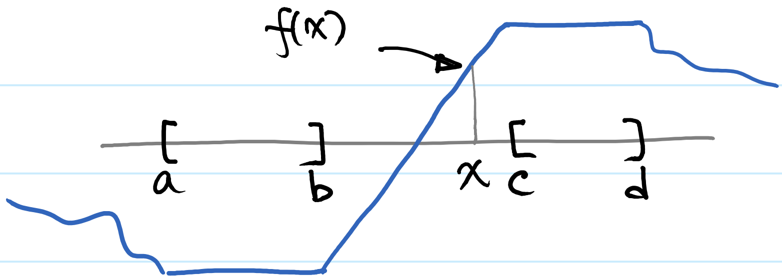
Lect09-0216-18-TietzX

Thursday, February 18, 2016 2:53 PM

Easy Question Given $[a,b], [c,d]$; $b < c$.

Find a continuous $f : \mathbb{R} \longrightarrow [-1,1]$ such that $f|_{[a,b]} \equiv -1$ and $f|_{[c,d]} \equiv 1$.



For $x \in [b,c]$, $f(x) = -1 + \frac{2}{c-b} \cdot (x-b)$

$$= \frac{(x-b)-(c-x)}{c-b}$$

Proposition. Let X be a metric space; $A, B \subset X$ be closed sets, $A \cap B = \phi$. Then $\exists$ continuous $f : X \longrightarrow [-1,1]$ such that

$$f|_A \equiv -1 \quad \text{and} \quad f|_B \equiv 1.$$

Qu Can we adopt the method of $X = \mathbb{R}$?

Note that in above, $a, d \in \mathbb{R}$ do not involve. $(x-b)$, $(c-x)$, $(c-b) = (x-b)+(c-x)$.

Idea of Proof.

For $x \notin A \cup B$, $f(x) = \dfrac{d(x,A) - d(x,B)}{d(x,A) + d(x,B)}$

where $d(x,S) = \inf\{d(x,y) : y \in S\}$ for any $S \subset X$.

Some necessary information,

① $d(x,S)=0 \Rightarrow x\in\bar{S}$

Using that both A, B are closed and $A\cap B=\phi$, we have denominator $\neq 0$

② $x \longmapsto d(x,S)$ is a continuous function

This guarantees the continuity of f.

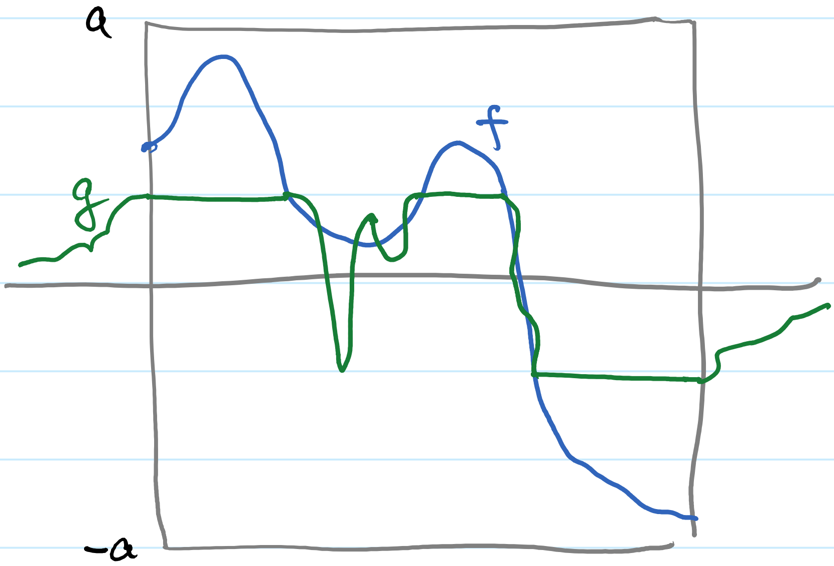
Urysohn Lemma. A normal topological space, the above proposition is satisfied.

Remark. Normal will be defined later.

The proof will be omitted in this course.

Tietz Extension Theorem Let X be a topological space satisfying the above proposition (e.g., a metric space or a normal space); and $F\subset X$ be closed. If $f: F\longrightarrow[-a,a]$ is continuous then $\exists$ continuous extension $\tilde{f}: X\longrightarrow[-a,a]$, i.e., $\tilde{f}|_F\equiv f$.

Let us construct a continuous function g on X from the given f. The figure serves a reference.



Take $A = f^{-1}[-a, \frac{-a}{3}]$, $B = f^{-1}[\frac{a}{3}, a]$

Then A, B are closed, $A \cap B = \phi$

By the given property of X (satisfying the proposition), $\exists$ continuous $g : X \longrightarrow [\frac{-a}{3}, \frac{a}{3}]$ such that $g|_A \equiv \frac{-a}{3}$, $g|_B \equiv \frac{a}{3}$.

From now, call this $g_1 \equiv g$.

Note that ① $\|g_1\| = \sup\{|g_1(x)| : x \in X\} \leq \frac{a}{3}$ on X

② $\|f - g_1\| = \sup\{|f(x) - g_1(x)| : x \in F\} \leq \frac{2a}{3}$ on F

Consider $f - g_1 : F \longrightarrow [\frac{-2a}{3}, \frac{2a}{3}]$ is continuous, by the same argument above, one has continuous

$$g_2 : X \longrightarrow [\frac{-a}{9}, \frac{a}{9}], \text{ i.e., } \|g_2\| \leq \frac{a}{9} \text{ on } X$$

and $f - g_1 - g_2 : F \longrightarrow [\frac{-4a}{9}, \frac{4a}{9}]$

Inductively, we have continuous functions

$$g_n : X \longrightarrow [\frac{-a}{3^n}, \frac{a}{3^n}] \quad \text{and}$$

$$f - \sum_{k=1}^{n} g_k : F \longrightarrow [-(\tfrac{2}{3})^n a, (\tfrac{2}{3})^n a]$$

By arguments similar to Math Analysis,

$$\sum_{k=1}^{n} g_k \xrightarrow{\text{uniformly}} \tilde{f} : X \longrightarrow [-a, a]$$

which is continuous

And $f \equiv \tilde{f}|_F$ as $\|f - \sum_{k=1}^{n} g_k\| \longrightarrow 0$ on F.