1a. let $I_k = \left[a + \frac{k(b-a)}{n}, \ a + \frac{(k+1)(b-a)}{n}\right]$

$$\min_{x \in I_k} x^2 = \left(a + \frac{k(b-a)}{n}\right)^2 \quad \text{as } a > 0$$

$$\Rightarrow L(P,f) = \sum_{k=0}^{n-1} \left(a + \frac{k(b-a)}{n}\right)^2 \left(\frac{b-a}{n}\right)$$

$$= \left(\frac{b-a}{n}\right)\left(\sum_{k=0}^{n-1}\left(a^2 + \frac{2a(b-a)}{n}k + \frac{(b-a)^2k^2}{n^2}\right)\right)$$

$$= \left(\frac{b-a}{n}\right)\left(a^2n + \left(\frac{2a(b-a)}{n}\right)\left(\frac{n(n-1)}{2}\right) + \frac{(b-a)^2}{n^2}\left(\frac{(n-1)(n)(2n-1)}{6}\right)\right)$$

$$= (b-a)a^2 + \frac{a(b-a)^2(n-1)}{n} + \frac{(b-a)^3(n-1)(2n-1)}{6n^2}$$

1b.
$$\lim_{|P| \to 0} L(P,f) = \lim_{n\to\infty} L(P,f)$$

$$= \lim_{n\to\infty}\left[(b-a)a^2 + \frac{a(b-a)^2(n-1)}{n} + \frac{(b-a)^3(n-1)(2n-1)}{6n^2}\right]$$

$$= (b-a)a^2 + a(b-a)^2 + \frac{(b-a)^3}{3}$$

$$= (b^3 - a^3)/3$$

2

$$f(x) = \arctan x$$

$$f'(x) = \frac{1}{1+x^2}$$

$\forall \varepsilon > 0, \quad \forall x \in \mathbb{R}, \quad \forall y \in (x-\varepsilon, x+\varepsilon)$

by mean value Thm, $\exists$ c between x and y

s.t. $|f(x) - f(y)| = f'(c)\,|x-y|$

$$\Rightarrow |f(x) - f(y)| = \frac{1}{1+c^2}|x-y|$$

$$\leq |x-y|$$

$$< \varepsilon$$

$\Rightarrow$ f is uni. cont.

3. ~~take~~ $\forall \delta > 0$, $\exists$ integer n s.t $\frac{1}{n} < \delta$

take $x_n = \frac{1}{n}$, $y_n = \frac{1}{2n}$ $\quad x_n, y_n \in (0,1)$

$|x_n - y_n| = \frac{1}{2n} < \delta$.

but $|f(x_n) - f(y_n)| = |n^2 - 4n^2| = 3n^2 \geq 3$

$\Rightarrow$ f is not uni. cont

4. $\forall \varepsilon > 0$.

$\forall x \in \mathbb{R}$

if $x \in (-\varepsilon/4, \varepsilon/4)$, $\forall y \in (x - \varepsilon/4, x + \varepsilon/4)$

$|f(x) - f(y)| \leq |x| + |y|$

$\leq \varepsilon/4 + \varepsilon/4 = \varepsilon/2 < \varepsilon$

if $|x| \geq \varepsilon/4$,

$f'(x) = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}$

~~$\forall y \in (x$~~ take $\eta = \min\{\varepsilon/8, \frac{\varepsilon^2}{\varepsilon + 8}\}$

$\forall y \in (x - \eta, x + \eta)$

$|y| \geq \varepsilon/8$ as $\eta \geq \varepsilon/8$.

by means value Thm, $\exists c$ between y and x

s.t $|f(x) - f(y)| = f'(c)|x - y|$

as $|c| \geq \varepsilon/8$.

$\Rightarrow |f(x) - f(y)| \leq (1 + \frac{8}{\varepsilon})\eta$

$\leq (1 + 8/\varepsilon)(\frac{\varepsilon^2}{\varepsilon + 8})$

$= \varepsilon$

in general, take $\delta = \min\{\varepsilon/4, \eta\}$ then

~~$\forall \varepsilon > 0$~~ $\forall y \in (x - \delta, x + \delta)$, $|f(x) - f(y)| < \varepsilon$

$\Rightarrow$ uni cont.

5a, obviously, $p_n(x) \to 0$

$\forall \varepsilon > 0$, $\forall x \in \mathbb{R}$

$$e^x = 1 + x + \frac{x^2}{2} + \dots$$

$$p_n(x) = \frac{nx}{1 + nx^2 + \frac{n^2x^4}{2} + \dots}$$

use the 3rd ~~ϕ~~ or higer order term to prove the convergence.

but we cannot get an estimate independant of x.

$\forall n$, take $x_n = \frac{1}{\sqrt{n}}$

$$p_n(x_n) = n\left(\frac{1}{\sqrt{n}}\right)e^{-1} = \sqrt{n}/e > 1/e$$

$\Rightarrow$ not uni conv.

5b. $\forall \varepsilon > 0$,

$\forall x \in [\frac{1}{2}, 1]$ $\quad \forall n > 4/\varepsilon$

$$|q_n(x) - 0| = \frac{nx}{1 + n^2x^2}$$

$$\leq \frac{n}{1 + n^2/4} = \frac{4n}{4 + n^2} \leq \frac{4}{n} < \varepsilon$$

$\Rightarrow$ uni. conv.

6. $\forall x \in (-1, 1)$,

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \qquad \sum_{k=0}^{\infty} x^{k+1} = \frac{x}{1-x}$$

$$\Rightarrow \sum_{k=0}^{\infty} x^k(1-x) = 1$$

if $x = 1$

$1 - x = 0 \quad \Rightarrow \quad \sum_{k=0}^{\infty} x^k(1-x) = 0$.

as $\sum_{k=0}^{n} x^k(1-x)$ is cont,

if it conv. uni, then $\sum^{\infty} x^k(1-x)$ is cont.

but
$$\sum^{\infty} x^k(1-x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } x = 1 \end{cases}$$

$\Rightarrow$ it is not uni. conv.