MATH 2055 Assmt 2 Suggested Solutions

1. The following is the idea:

$$\left|\frac{n^2+n+1}{n^3}-0\right| = \frac{n^2+n+1}{n^3} < \frac{C}{n} \qquad (1)$$

we want to find a positive constant C so that this inequality is valid.

(Reason: when $n >> 1$, $\frac{n^2+n+1}{n^3} \approx \frac{1}{n}$).

To find this constant, we consider

$$(n^2+n+1)n < C \cdot n^3 \quad (*)$$

$$\Leftrightarrow \quad n^3+n^2+n < Cn^3$$

$$\Leftrightarrow \quad n^2+n < (C-1)n^3 \quad (**)$$ (i.e. C must be bigger than 1 !)

Let's try and see whether $C=2$ works or not.

Choosing $C=2$ in $(**)$ gives

$$n^2+n < n^3$$

which motivates us to check whether this inequality is valid or not.

BUT,

$$n^2+n < n^3 \Leftrightarrow n+1 < n^2$$

$$\Leftrightarrow \quad 1 < n^2-n$$

$$\Leftrightarrow \quad 1 < n(n-1)$$

which is true if $n>1$.

From the above discussion, we see that

If $n > 1$, then $(n^2+n+1)\, n < 2n^3$ in (*)

$$\Rightarrow \quad \frac{n^2+n+1}{n} < \frac{2}{n}$$

Therefore if for ANY given $\varepsilon > 0$, we choose

$n =$ the next natural no. bigger than or equal to $\frac{2}{\varepsilon}$,

then $$n \geq \frac{2}{\varepsilon} \iff \varepsilon \geq \frac{2}{n} > \left| \frac{n^2+n+1}{n^3} - 0 \right|$$

Hence $$\lim_{n \to \infty} \frac{n^2+n+1}{n^3} = 0.$$ (of course, we require also that $n > 1$!)

Comment: $N = \max\{1, \text{ next nat. no.} \geq \frac{2}{\varepsilon}\}$.

2. $$\left|\frac{2^n+3}{2^n+n+10}-1\right| = \left|\frac{\cancel{2^n}+3-\cancel{2^n}-n-10}{2^n+n+10}\right|$$

$$= \frac{|-n-7|}{|2^n+n+10|} = \frac{n+7}{2^n+n+10} \overset{(*)}{<} \frac{8n}{2^n}$$

If $n>1$ $\big(\because\ n>1 \iff 7n > 7$

$\iff 7n+n > n+7$

$\iff 8n > n+7\big)$

Summary: If $n>1$ then $\left|\frac{2^n+3}{2^n+n+10}-1\right| < \frac{8n}{2^n}$

This motivates to find (for each given $\varepsilon>0$), N such that

$\frac{8n}{2^n} < \varepsilon$. How to get this N?

IDEA: $$2^n = (1+1)^n = 1 + C_1^n\cdot 1 + C_2^n \cdot 1 + C_3^n\cdot 1 + \cdots$$

$$= 1 + n + \frac{n(n-1)}{2} + \cdots$$

$$> \frac{n(n-1)}{2}$$

$$\Rightarrow \frac{8n}{2^n} < \frac{8n}{\frac{n(n-1)}{2}} = \frac{16\cancel{n}}{\cancel{n}(n-1)}$$

Hence it is O.K. if we can find N satisfying

$$\frac{16}{n-1} < \varepsilon \quad (**) \qquad (\text{for any given } \varepsilon > 0).$$

(**) leads to $n-1 > \frac{16}{\varepsilon} \iff n > \frac{16}{\varepsilon}+1$

Hence we can let

$$N = \text{the next nat. no.} \geq \frac{16}{\varepsilon}+1.$$

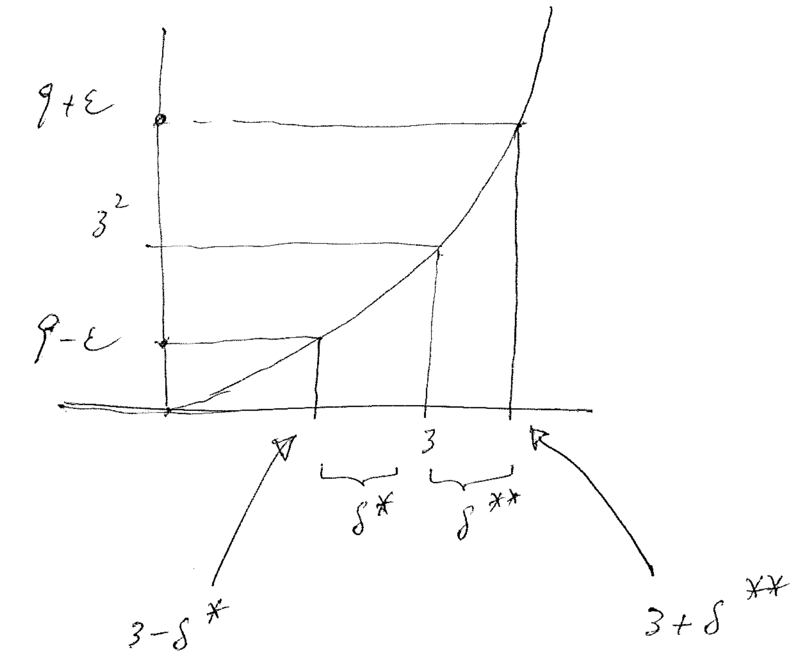
But recalling that we have assumed $n \geq 1$,

therefore need to let

$$N = \max\left\{ \text{the next nat. no.} \geq \frac{16}{\varepsilon}+1, \quad 1 \right\}$$

\#

3. (a)



From $9-\varepsilon < x^2 < 9+\varepsilon$, we are led to consider

the point x_1 which satisfies $f(x_1) = x_1^2 = 9-\varepsilon$ —— ①

and the point x_2 which satisfies $f(x_2) = x_2^2 = 9+\varepsilon$ —— ②

① $\Rightarrow$ $x_1 = \sqrt{9-\varepsilon}$ $(\varepsilon < 9\ !!)$

But $x_1 = 3 - \delta^* = \sqrt{9-\varepsilon}$

$\Rightarrow$ $\delta^* = 3 - \sqrt{9-\varepsilon}$

Similarly $x_2^2 = 9+\varepsilon \Rightarrow x_2 = \sqrt{9+\varepsilon}$

But $x_2 = 3 + \delta^{**}$ $\Rightarrow$ $\delta^{**} = x_2 - 3 = \sqrt{9+\varepsilon} - 3$.

ANS) $\delta^* =$ distance between 3 & $\sqrt{9-\varepsilon}$

$\delta^{**} =$ distance between $\sqrt{9+\varepsilon}$ & 3.

(b). $|x-3| < \max\{3-\sqrt{9-\varepsilon},\ \sqrt{9+\varepsilon}-3\}$

(or <u>BOTH</u> $\begin{cases} 3-\sqrt{9-\varepsilon} \\ \sqrt{9+\varepsilon}-3 \end{cases}$)

$$\Rightarrow\ -(3-\sqrt{9-\varepsilon}) < x-3 < 3-\sqrt{9-\varepsilon} \quad \text{—— (1)}$$

<u>and</u>

$$-(\sqrt{9+\varepsilon}-3) < x-3 < \sqrt{9+\varepsilon}-3 \quad \text{—— (2)}$$

$\Rightarrow$ From (1) $\quad +\sqrt{9-\varepsilon} < x$

and

From (2) $\quad x < \sqrt{9+\varepsilon}$

$$\Rightarrow\ \underset{\substack{\vee\\0}}{\sqrt{9-\varepsilon}} < x < \underset{\substack{\vee\\0}}{\sqrt{9+\varepsilon}} \quad (1)$$

$$\Rightarrow\ 9-\varepsilon < x^2 < 9+\varepsilon$$

$$\Rightarrow\ -\varepsilon < x^2-9 < \varepsilon$$

$$\Rightarrow\ |x^2-9| < \varepsilon$$

<u>Footnote</u> (1). We are using here

If $0<A$, $0<B$, then $A<B \iff A^2<B^2$

Proof: Since $0<A<B$ hence $\begin{cases} B-A>0 \\ B+A>0 \end{cases}$

$\iff (B+A)(B-A)>0$

$\iff B^2-A^2>0$

(c) $\delta^* = 3 - \sqrt{9-\varepsilon}$, $\delta^{**} = \sqrt{9+\varepsilon} - 3$

$$\delta^* > \delta^{**} \iff \cancel{\delta\delta^*}\ 3 - \sqrt{9-\varepsilon} > \sqrt{9+\varepsilon} - 3$$

$$\iff 6 > \sqrt{9+\varepsilon} + \sqrt{9-\varepsilon}$$

$$\iff 36 > 9 + \cancel{\varepsilon} + 9 - \cancel{\varepsilon} + 2\sqrt{9+\varepsilon}\sqrt{9-\varepsilon}$$

(Using Footnote (1) on p.5)

$$\iff \cancel{18}^{\,9} > \cancel{2}\sqrt{9^2-\varepsilon^2}$$

$$\iff 81 > 81 - \varepsilon^2$$

$$\iff 0 > -\varepsilon^2$$

$$\iff \varepsilon^2 > 0 .$$

(a) $0 < |x-3| < \delta_2 = 1$

triangle inequality (i.e. $|a+b| \leq |a|+|b|$)

$$\Rightarrow |x+3| = |x+6-3| \leq |x-3| + |6| = |x-3| + 6 < \delta_2 + 6 = 7$$

(b). $|x^2 - 3^2| = |x-3|\,|x+3|$ $\leq 7\delta_2 < \varepsilon$ goal!

Letting $\varepsilon \geq 7\delta_2$, we get $\delta_2 \leq \frac{\varepsilon}{7}$ (under the assumption $\delta_2 = 1$)

Hence if $\delta_2 = \min\{\frac{\varepsilon}{7}, 1\}$ then

If $\quad 0 < |x-3| < \delta_2$

then $\quad |x^2 - 3^2| \leq 7\delta_2 \leq \varepsilon \quad$ #

(c) $\quad \delta_1$ of Method (I) is given by

$$\delta_1 = \min\{\delta^*, \delta^{**}\}$$

$$= \delta^{**} = \sqrt{9+\varepsilon} - 3.$$

Therefore what we need to show is:

$$\sqrt{9+\varepsilon} - 3 < \delta_2 = \min\left\{1, \frac{\varepsilon}{7}\right\}$$

$$\Longleftrightarrow \quad \sqrt{9+\varepsilon} - 3 < \begin{cases} 1 & \text{—— (1)} \\ \frac{\varepsilon}{7} & \text{—— (2)} \end{cases}$$

$\Longleftrightarrow$

From (1) $\quad \sqrt{9+\varepsilon} < 4 \Longleftrightarrow 9+\varepsilon < 16 \Longleftrightarrow \varepsilon < 7$

From (2) $\quad \sqrt{9+\varepsilon} - 3 < \frac{\varepsilon}{7} \Longleftrightarrow \sqrt{9+\varepsilon} < 3 + \frac{\varepsilon}{7}$

$$\Longleftrightarrow \cancel{9} + \varepsilon < \cancel{9} + \frac{\varepsilon^2}{7^2} + \frac{6\varepsilon}{7}$$

$$\Longleftrightarrow \quad 0 < \frac{\varepsilon^2}{7^2} - \frac{\varepsilon}{7}$$

$$\Longleftrightarrow \quad 0 < \frac{\varepsilon}{7}\left(\frac{\varepsilon}{7} - 1\right)$$

$$\Longleftrightarrow \quad \varepsilon < 0 \text{ or } \varepsilon > 7$$

Hence there isn't any such $\varepsilon > 0$!

In[21]:=
```
Plot[{Sqrt[9 + y] - 3, Min[1, y / 7]},
 {y, 0, 30}, AspectRatio → 0.7, PlotRange → {0, 1.3},
 PlotStyle → {{Blue, Thickness[0.003]}, {Red, Thickness[0.003], Dashed}}]
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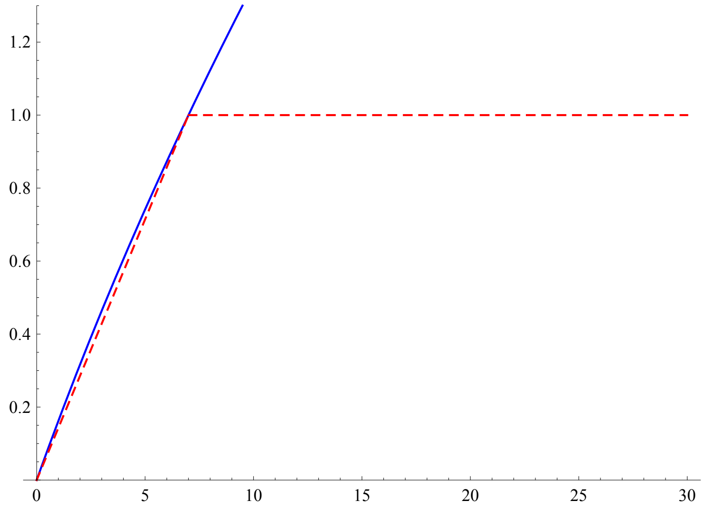
Out[21]=



4 (a) $\lim\limits_{h\to 0} \dfrac{f(2+h)-f(2)}{h}$ doesn't exist.

or equivalently

$$\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} \neq \lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h}$$

or (using the $\varepsilon-\delta$ definition).

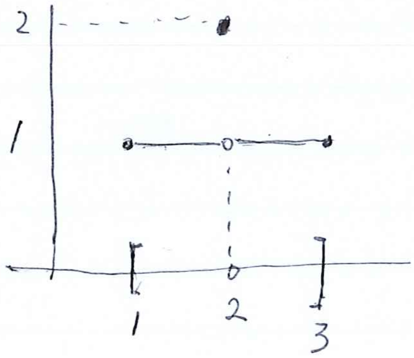
$$\exists \varepsilon > 0 \ \forall \delta > 0 \ \mid \ \exists x \quad 0 < |x-2| < \delta \ \text{ and } \ \left| \frac{f(x)-f(2)}{x-2} \right| \geq \varepsilon$$

($x \in [1,3]$ of course!)

(b) c is an absolute max. point of f if

$$\forall x \in [1,3] \ \mid \ f(x) \leq f(c).$$

(c) No.

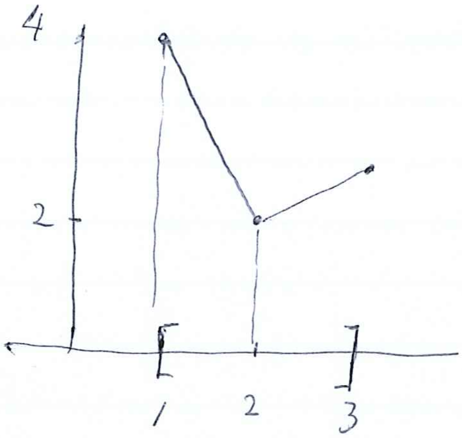


«Counter-example» $f(x) = \begin{cases} 1, & \text{if } 1 \leq x < 2 \text{ or } 2 < x \leq 3 \\ 2, & \text{if } x = 2. \end{cases}$

then $f(2) \geq f(x) \quad \forall x \in [1,3]$

But f is not cont. at $x = 2$.

(d)



Comment: I haven't written down the formulas for the function here, but it can be done!

5. Let $f(x) = x^3 - \sin(100x) - 1000$

Let $N = (1002)^{\frac{1}{3}}$

then $f(N) = (1002)^{\frac{1}{3}\cdot 3} - \sin\left(100 \times (1002)^{\frac{1}{3}}\right) - 1000$

$\geq 1002 - 1 - 1000$ ($\because -1 \leq \sin x \leq +1$)

$= 1 > 0$ ($\therefore -1 \leq -\sin x \leq +1$)

Hence $f(N) > 0$.

On the other hand, $f(0) = 0^3 - \sin(100 \times 0) - 1000$

$= -1000 < 0$.

Hence by the Intermediate Value Theorem ($\because$ f is a cont. fn.), $\exists\ \xi$ in $[0, N]$ such that

$f(\xi) = 0$.

6. (a) $|x_n| = \left|(-1)^n \frac{2^n-1}{2^n+1}\right| = \left|\frac{2^n-1}{2^n+1}\right| = \frac{2^n-1}{2^n+1}$

$(\because\ 2^n-1 \geq 0,\ 2^n+1 > 0)$

But $\frac{2^n-1}{2^n+1} = \frac{2^n+1-2}{2^n+1} = 1 - \frac{2}{2^n+1} < 1$

therefore $|x_n| < 1 \iff -1 < x_n < 1$

(b). If n is even, i.e. $n'_k := 2k$, then

$$x_{n'_k} = \frac{2^{2k}-1}{2^{2k}+1} = \frac{\frac{2^{2k}}{2^{2k}} - \frac{1}{2^{2k}}}{\frac{2^{2k}}{2^{2k}} + \frac{1}{2^{2k}}} = \frac{1-\frac{1}{2^{2k}}}{1+\frac{1}{2^{2k}}}$$

Hence $\lim_{k\to\infty} x_{n'_k} = \lim_{k\to\infty} \frac{1-\frac{1}{2^{2k}}}{1+\frac{1}{2^{2k}}} = 1.$

If n is odd, i.e. $n''_k = 2k+1$, then

$$x_{n''_k} = (-1)^{2k+1} \frac{2^{2k+1}-1}{2^{2k+1}+1} = -1 \cdot \frac{2^{2k+1}-1}{2^{2k+1}+1}$$

$$= -\left(\frac{1-\frac{1}{2^{2k+1}}}{1+\frac{1}{2^{2k+1}}}\right)$$

Hence $\lim_{k\to\infty} x_{n''_k} = \lim_{k\to\infty} -\left(\frac{1-\frac{1}{2^{2k+1}}}{1+\frac{1}{2^{2k+1}}}\right) = -1$ #