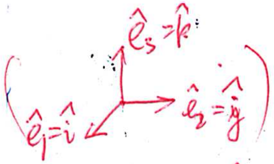
Differential Forms



Basic 1-form on $\mathbb{R}^n$: $dx_1, \cdots dx_n$ vs. $\hat{e}_1, \cdots, \hat{e}_n$

formal symbol

| in space gen by basic 1-form

General 1-form: $\omega = f_1 dx_1 + \cdots + f_n dx_n$ $\quad \vec{v} = v_1\hat{e}_1 + \cdots + v_n\hat{e}_n$

$f_1, \cdots, f_n$ smooth f's $\quad$ v_i smooth fns

"Vector fields"

Basic 2-form: $dx_i \wedge dx_j \quad 1 \le i < j \le n$ ("wedge")

General 2-form: $\omega = \sum_{i<j} f_{ij}\, dx_i \wedge dx_j$

f_{ij} smooth f^n.

How many basic forms? $\binom{n}{2} = \frac{n(n-1)}{2}$

Operation on differential forms

(1) addition. "Vector space"

$$\omega = f_1 dx_1 + \cdots + f_n dx_n$$

$$\eta = g_1 dx_1 + \cdots + g_n dx_n$$

$$\omega + \eta = (f_1+g_1)dx_1 + \cdots + (f_n+g_n)dx_n$$

Notation: $\Omega^1(\mathbb{R}^n) =$ ~~space of~~ $\{$(general) 1-form$\}$

Vector space with $C^\infty(\mathbb{R}^n)$-coeff. of dim $= n$. basis $= dx_1, \cdots, dx_n$

(2) **Wedge product**. $dx_i \wedge dx_j = -dx_j \wedge dx_i$.

Anti-commutative

$\Rightarrow dx_i \wedge dx_i = 0$.

Vectors $\vec{u}, \vec{v} \in$ "cross product" $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$

$\Omega^2(\mathbb{R}^n)$ Vector space of dim $= \binom{n}{2} = \frac{n(n-1)}{2}$.

Ex: $\omega = f_1 dx + f_2 dy + f_3 dz$

$\eta = g_1 dx + g_2 dy + g_3 dz$. $\qquad // -\eta\wedge\omega$

$$\omega\wedge\eta = (f_1g_2 - f_2g_1)\,dx\wedge dy + (f_1g_3 - f_3g_1)\,dx\wedge dz + (f_2g_3 - f_3g_2)\,dy\wedge dz$$

"$\hat{k}$" $\qquad \updownarrow \qquad$ "$-\hat{j}$" $\qquad$ "$\hat{i}$"

($dx \leftrightarrow \hat{i}$, $dy \leftrightarrow \hat{j}$, $dz \leftrightarrow \hat{k}$. ~~then~~

$\langle f_1, f_2, f_3\rangle \times \langle g_1, g_2, g_3\rangle = \langle f_2g_3 - f_3g_2,\ f_3g_1 - f_1g_3,\ f_1g_2 - f_2g_1\rangle$ Cross product!

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(3). exterior derivative d

rule: · $df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n$, "gradient" "only" acts on f"

· $d(dx_i) = 0$.

Example: $\omega = Pdx + Qdy + Rdz \in \Omega^1(\mathbb{R}^3)$.

$$d\omega = d(Pdx + Qdy + Rdz)$$
$$= dP \wedge dx + dQ \wedge dy + dR \wedge dz.$$

$dP = P_x dx + P_y dy + P_z dz$. etc.

Plug in and use ~~the~~ def of wedge product $\wedge$

$$= (Q_x - P_y)\underbrace{dx\wedge dy}_{\hat{k}} + (R_x - P_z)\underbrace{dx\wedge dz}_{-\hat{j}} + (R_y - Q_z)\underbrace{dy\wedge dz}_{\hat{i}}.$$

" $\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$. Curl!

Def: Basic k-form. $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ $\quad 1 \le i_1 < \cdots < i_k < n$. No repetition!

$$\left(\cdots dx \wedge dy \wedge \cdots \wedge dx \wedge \cdots = (-1)^{\alpha} \cdots dx \wedge dx \wedge \cdots = (-1)^{\alpha+1} \cdots dx \wedge dx \wedge \cdots = 0\right)$$

General k-form: $\sum_{1 \le i_1 < \cdots < i_k \le n} f_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

$$\dim \Omega^k(\mathbb{R}^n) = \#\text{basic } k\text{-forms} = \binom{n}{k}.$$

Example: $dx\wedge dy \wedge dz = -dy \wedge dx \wedge dz = dy \wedge dz \wedge dx = -dz \wedge dy \wedge dx$
$= dz \wedge dx \wedge dy = -dx \wedge dz \wedge dy$.

Generally, $dx_{i_1} \wedge \cdots \wedge dx_{i_k} = (-1)^{\sigma} dx_{\sigma(i_1)} \wedge dx_{\sigma(i_2)} \cdots \wedge dx_{\sigma(i_k)}$

σ: permutation of $\{i_1, \dots, i_k\}$ $\quad (-1)^{\sigma} = \begin{cases} 1 & \text{even \# transp.} \\ -1 & \text{odd \# } \cdots \end{cases}$.

Example: $\omega = P\underbrace{dy \wedge dz}_{\hat{i}} - Q\underbrace{dx \wedge dz}_{-\hat{j}} + R\underbrace{dx \wedge dy}_{\hat{k}}$. $\langle P, Q, R \rangle$.

$$d\omega = dP \wedge dy \wedge dz - dQ \wedge dx \wedge dz + dR \wedge dx \wedge dy$$
$$= \underbrace{(P_x + Q_y + R_z)}_{\text{divergence!}} dx \wedge dy \wedge dz.$$

Example: $\omega = P\,dy\wedge dz - Q\,dx\wedge dz + R\,dx\wedge dy$

$\eta = L\,dx + M\,dy + N\,dz$

$$\omega\wedge\eta = (PL + QM + RN)\,dx\wedge dy\wedge dz = \eta\wedge\omega.$$

dot product !

Summerize

- $\Omega^k(\mathbb{R}^n)$. $\dim = \binom{n}{k}$. $0\le k\le n$; $\Omega^k(U)$ for $U\subset\mathbb{R}^n$

  In particular: 0-form: smooth f^n f. $\Omega^0 \cong \Omega^n \cong C^\infty$

  n-form: $f\,dx_1\wedge\cdots\wedge dx_n$

- Addition: k-form $+$ k-form $=$ k-form.
- Wedge product. $\Omega^k(\mathbb{R}^n)\wedge\Omega^l(\mathbb{R}^n)\longrightarrow\Omega^{k+l}(\mathbb{R}^n)$.
- exterior derivative d. $\Omega^k(\mathbb{R}^n)\xrightarrow{d}\Omega^{k+1}(\mathbb{R}^n)$.

General properties:

Thm. $\omega\in\Omega^k(\mathbb{R}^n)$, $\eta\in\Omega^l(\mathbb{R}^n)$.

then $\omega\wedge\eta = (-1)^{kl}\eta\wedge\omega$.

(Pf: $\omega = f\,dx_{i_1}\wedge\cdots\wedge dx_{i_k}$, $\eta = g\,dx_{j_1}\wedge\cdots\wedge dx_{j_l}$,

$\omega\wedge\eta = (f\cdot g)\,dx_{i_1}\wedge\cdots\wedge dx_{i_k}\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_l}$

$\eta\wedge\omega = (f\cdot g)\,dx_{j_1}\wedge\cdots\wedge dx_{j_l}\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k}$

total $k\cdot l$ ~~trans~~position.).

Thm. $\omega\in\Omega^k(\mathbb{R}^n)$. $\eta\in\Omega^l(\mathbb{R}^n)$ ("Leibnitz rule")

then $d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^k\omega\wedge d\eta$

(Pf: $\omega = f\,dx_{i_1}\wedge\cdots\wedge dx_{i_k}$, $\eta = g\,dx_{j_1}\wedge\cdots\wedge dx_{j_l}$, $I\cap J = \phi$, $\cdots$)

Thm $d^2\omega = 0$ for all $\omega \in \Omega^k(\mathbb{R}^n)$.

· Special case: $\omega = f \in \Omega^0(\mathbb{R}^n)$

$$df = \frac{\partial f}{\partial x_1}dx_1 + \cdots + \frac{\partial f}{\partial x_n}dx_n = \sum_i \frac{\partial f}{\partial x_i}dx_i$$

$$d(df) = \sum_{i,j} \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) dx_j \wedge dx_i$$

$$= \sum_{1\le i<j\le n} \cancel{\left(\frac{\partial^2 f}{\partial x_i\partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i}\right)} dx_i\wedge dx_j = 0$$

mixed derivative.

· In General, $\omega = f\, dx_{i_1}\wedge\cdots\wedge dx_{i_k}$

$$d\omega = df\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k}.$$

$$d(d\omega) = \underbrace{ddf}_{=0}\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k} - df\wedge \underline{d(dx_{i_1}\wedge\cdots\wedge dx_{i_k})}$$

$$\underbrace{ddx_{i_1}}_{=0}\wedge\cdots\wedge dx_{i_k} - dx_{i_1}\wedge \underline{d(\cdots)}$$

inductively.

Can also be proved directly using mixed derivative.

□

Def: $\omega\in\Omega^k(\mathbb{R}^n)$, $\Omega^k(U)$: closed form if $d\omega=0$

exact form if $\omega = d\eta$ for some $\eta\in\Omega^{k-1}(\mathbb{R}^n)$

exact form $\Longrightarrow$ closed form.

$(d\omega = dd\eta = 0)$.

$\overset{?}{\Longleftarrow}$

(Example: $U = \mathbb{R}^2-\{0\}$, $\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \in \Omega^1(U)$.

closed, but not exact)

Example: · $\omega = f\in\Omega^0(\mathbb{R}^n)$ closed $\Leftrightarrow df=0$. $\Leftrightarrow f$ const

· $\omega = Pdx+Qdy+Rdz\in\Omega^1(\mathbb{R}^3)$. closed

$\Leftrightarrow d\omega = (Q_x-P_y)dx\wedge dy + (R_x-P_z)dx\wedge dz + (R_y-Q_z)dy\wedge dz = 0$.

$\Leftrightarrow \boxed{\mathrm{curl}\langle P,Q,R\rangle = 0}$.

· ω exact.

$\Leftrightarrow \omega = df = f_x\, dx + f_y\, dy + f_z\, dz$.

$\Leftrightarrow \langle P,Q,R\rangle$ $\boxed{\text{gradient field}}$.

In $\mathbb{R}^3$.

Differential Forms		Vector Calculus
0-form Ω^0	$\xleftrightarrow{=}$	Scalar (function)
$\downarrow d$ $\quad f \xrightarrow{d} f_x dx + f_y dy + f_z dz$		$\downarrow \nabla$ $\quad f \xrightarrow{\nabla} \langle f_x, f_y, f_z \rangle = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$
1-form Ω^1	$\xleftrightarrow{dx \leftrightarrow \hat{i},\ dy \leftrightarrow \hat{j},\ dz \leftrightarrow \hat{k}}$	Vector field $\vec{F} = \langle P, Q, R \rangle$
$\downarrow d$ $\quad P dx + Q dy + R dz \xrightarrow{d} (R_y - Q_z) dy \wedge dz + (P_z - R_x) dz \wedge dx + (Q_x - P_y) dx \wedge dy$		$\downarrow$ curl $\nabla \times$ $\quad \xrightarrow{\nabla \times} \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$
2-form Ω^2	$\xleftrightarrow{dy \wedge dz \leftrightarrow \hat{i},\ dz \wedge dx \leftrightarrow \hat{j},\ dx \wedge dy \leftrightarrow \hat{k}}$	Vector field $\vec{F} = \langle P, Q, R \rangle$
$\downarrow d$ $\quad P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy \xrightarrow{d} (P_x + Q_y + R_z) dx \wedge dy \wedge dz$		$\downarrow$ divergence $\nabla \cdot$ $\quad \xrightarrow{\nabla \cdot} P_x + Q_y + R_z$
3-form Ω^3	$\xleftrightarrow{dx \wedge dy \wedge dz \leftrightarrow 1}$	Scalar (function)

Rmk: • $d^2 = 0$ $\Leftrightarrow$ $\text{curl}(\nabla f) = 0$; $\text{div}(\text{curl}\,\vec{F}) = 0$.

• closed form $\not\Rightarrow$ exact form. $\Leftrightarrow$ $\exists \vec{F}$ s.t. $\text{curl}\,\vec{F} = 0$ but $\vec{F}$ not gradient, etc.

"Pull-back" of differential forms.

$U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$

A smooth map $\varphi : U \longrightarrow V$

$(x_1, \cdots, x_m) \rightsquigarrow (y_1 = \varphi_1(x_1, \cdots, x_m), \cdots, y_n = \varphi_n(x_1, \cdots, x_m))$.

$\omega \in \Omega^k(V)$.

Define: $\underline{\underline{\varphi^* \omega}} \in \Omega^k(U)$ "pull-back".

- If $\omega = f \in C^\infty(V) = \Omega^0(V)$, Let $\boxed{\varphi^* \omega = f \circ \varphi} \in C^\infty(U) = \Omega^0(U)$
- If $\omega = dy_i \in \Omega^1(V)$,

  Let $\boxed{\varphi^* \omega = d\underbrace{\varphi_i(x_1, \cdots, x_m)}_{\in C^\infty(U)} = \sum_{j=1}^{m} \frac{\partial \varphi_i}{\partial x_j} dx_j} \in \Omega^1(U)$.
- In general, $\omega = \sum f_{i_1 \cdots i_k} dy_{i_1} \wedge \cdots \wedge dy_{i_k} \in \Omega^k(U)$.

  Let $\varphi^* \omega := \sum \varphi^*(f_{i_1 \cdots i_k}) \varphi^*(dy_{i_1}) \wedge \cdots \wedge \varphi^*(dy_{i_k}) = \sum g_{j_1 \cdots j_k} dx_{j_1} \wedge \cdots \wedge dx_{j_k} \in \Omega^k(U)$

Example. Let $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(\rho, \phi, \theta) \rightsquigarrow (\overset{x}{\overset{\shortparallel}{\rho \sin\phi \cos\theta}}, \overset{y}{\overset{\shortparallel}{\rho \sin\phi \sin\theta}}, \overset{z}{\overset{\shortparallel}{\rho \cos\phi}})$.

$\omega = dx \wedge dy \wedge dz \in \Omega^3(\mathbb{R}^3)$

Then
$$\varphi^* \omega = \overbrace{(\sin\phi \cos\theta\, d\rho + \rho \cos\phi \cos\theta\, d\phi - \rho \sin\phi \sin\theta\, d\theta)}^{\varphi^* dx}$$
$$\wedge \overbrace{(\sin\phi \sin\theta\, d\rho + \rho \cos\phi \sin\theta\, d\phi + \rho \sin\phi \cos\theta\, d\theta)}^{\varphi^* dy}$$
$$\wedge \overbrace{(\cos\phi\, d\rho - \rho \sin\phi\, d\phi)}^{\varphi^* dz}$$
$$= (\rho^2 \sin^3\phi + \rho^2 \sin\phi \cos^2\phi)\, d\rho \wedge d\phi \wedge d\theta = \boxed{\rho^2 \sin\phi\, d\rho \wedge d\phi \wedge d\theta}$$

In general, $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$(x_1, \cdots, x_n) \rightsquigarrow (y_1(x_1, \cdots, x_n), \cdots, y_n(x_1, \cdots, x_n))$.

Suppose $\omega = dy_1 \wedge \cdots \wedge dy_n \in \Omega^n(\mathbb{R}^n)$

then
$$\varphi^* \omega = \left(\sum_j \frac{\partial y_1}{\partial x_j} dx_j\right) \wedge \cdots \wedge \left(\sum_j \frac{\partial y_n}{\partial x_j} dx_j\right).$$
$$= \det \underbrace{\left[\frac{\partial(y_1, \cdots, y_n)}{\partial(x_1, \cdots, x_n)}\right]}_{\text{Jacobian matrix!}} dx_1 \wedge \cdots \wedge dx_n.$$

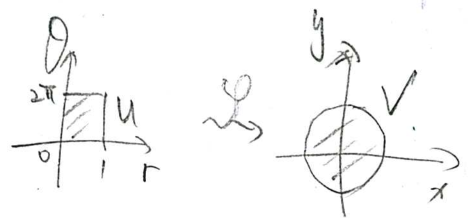
Integration of forms

Suppose $R \subset \mathbb{R}^n$, $\omega \in \Omega^n(R)$.
$\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$
$\qquad\qquad\qquad\qquad\qquad f\,dx_1 \wedge \cdots \wedge dx_n$.

Define. $\boxed{\displaystyle\int_R \omega \underset{(\pm)}{:=} \iint_R \cdots \int f \underbrace{dx_1 \cdots dx_n}_{\text{"}dV\text{"}}}$

Rmk: R "oriented domain", $f\,dx_1 \wedge \cdots \wedge dx_n = -f\,dx_2 \wedge dx_1 \wedge \cdots \wedge dx_n$

Change of Variable:



$\varphi: U \subseteq \mathbb{R}^n \longrightarrow V \subseteq \mathbb{R}^n$.
$(x_1, \cdots, x_n) \rightsquigarrow (y_1, \cdots, y_n)$.

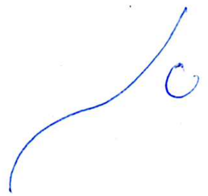
We learnt Formula: $\displaystyle\iint_V \cdots \int f(y_1 \cdots y_n)\,dy_1 \cdots dy_n = \iint_U \cdots \int f\circ\varphi(x_1, \cdots, x_n) \left|\frac{\partial(y_1, \cdots, y_n)}{\partial(x_1, \cdots, x_n)}\right| dx_1 \cdots dx_n$

In differential form: $\boxed{\displaystyle\int_V \omega \underset{(\pm)}{=} \int_{U=\varphi^* V} \varphi^* \omega}$. $\quad \omega = f\,dx_1 \wedge \cdots \wedge dx_n$

Rmk: Orientation preserving v.s. reversing.

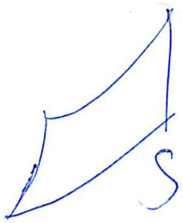
More generally, M: k-dim subspace in $\mathbb{R}^n$.
(manifold, k-dim "surface").

Integration over M.



- $k=1$. Line integral $\int_C f\,ds$

  $\int_C \vec{F} \cdot d\vec{r}$.



- $k=2$. $n=3$. Surface integral. $\iint_S f \cdot dS$

  $\iint_S \vec{F} \cdot \hat{n}\,dS$

Differential form: $\omega \in \Omega^k(M)$

One can define $\int_M \omega$:

Take a "parametrization" $\varphi: U \subseteq \mathbb{R}^k \longrightarrow M \subseteq \mathbb{R}^n$.

Let $\boxed{\displaystyle\int_M \omega := \int_{U=\varphi^* M} \underbrace{\varphi^* \omega}_{g\,dx_1 \wedge \cdots \wedge dx_k}}$ $\left(= \int_U g\,dx_1 \cdots dx_k \right)$

Example (1) Recall $\vec{F} = \langle P, Q, R \rangle$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz.$$

On the other hand, if we let $\omega = P\,dx + Q\,dy + R\,dz \in \Omega^1$

Let $\varphi : [a,b] \subseteq \mathbb{R} \longrightarrow C \subseteq \mathbb{R}^3$

$t \longmapsto (x(t), y(t), z(t))$.

Then $\displaystyle\int_C \omega = \int_{[a,b]} \varphi^*\omega = \int_{[a,b]} \big(P\cdot x'(t) + Q\cdot y'(t) + R\, z'(t)\big)\,dt$

(2). $\vec{F} = \langle P, Q, R \rangle$. Suppose $S : \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$.

$$\iint_S \vec{F}\cdot \hat{n}\, dS$$

$$= \iint_S \langle P, Q, R \rangle \cdot \underbrace{(\vec{r}_u \times \vec{r}_v)}_{= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}}\, du\, dv$$

$$= \iint_S \left(P\cdot \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} - Q\cdot \begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix} + R \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \right) du\, dv$$

In differential form.

Let $\omega = P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy$.

__Claim__. $\displaystyle\int_S \omega$ is the above!

__Pf:__ Let $U \subset \mathbb{R}^2 \xrightarrow{\varphi} S \subset \mathbb{R}^3$

$(u, v) \rightsquigarrow (x(u,v), y(u,v), z(u,v))$.

Then $\displaystyle\int_S \omega = \int_U \varphi^*\omega = \int_U \big(P|\ \ | - Q|\ \ | + R|\ \ |\big)\, du\wedge dv$.

Note: $dy\wedge dz = (y_u du + y_v dv)\wedge(z_u du + z_v dv) = \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} du\wedge dv$

$dz\wedge dx = \begin{vmatrix} z_u & x_u \\ z_v & x_v \end{vmatrix} du\wedge dv$.

$dx\wedge dy = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} du\wedge dv$.

__Rmk:__ $\boxed{\hat{n}\,dS = (\vec{r}_u \times \vec{r}_v)\, du\, dv \longleftrightarrow \langle dy\wedge dz, dz\wedge dx, dx\wedge dy \rangle}$

(v.s. $\hat{n}\,ds = \langle dy, -dx \rangle$).

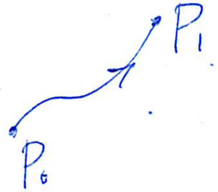
Generalized Stokes' theorem

M: k-dim mfld, possibly with boundary; ∂M $(k-1)$-dim

$\omega \in \Omega^{k-1}(M)$, $\quad d\omega \in \Omega^{k}(M)$.

$$\boxed{\int_M d\omega = \int_{\partial M} \omega.}$$

Example

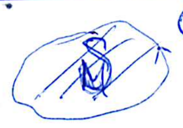
1) $M =$ curve C 

$\quad \partial M = P_1 - P_0$

$\omega = f \in \Omega^0(C)$. $\quad d\omega = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$

Stokes: $\int_M d\omega = \int_C \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n = \int_C \nabla f \cdot d\vec{r}$

$= \int_{\partial M} \omega = f(P_1) - f(P_0)$. $\quad$ Fund. Thm of ~~Calc~~ Line integral.

2) $M =$ 

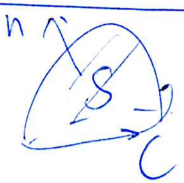
$\subset \mathbb{R}^2$. $\quad \partial M = C$.

$\omega = P\,dx + Q\,dy$. $\quad d\omega = (Q_x - P_y)\, dx \wedge dy$.

Stokes: $\int_M d\omega = \int_M (Q_x - P_y)\, dx \wedge dy = \iint_S \operatorname{curl} \vec{F}\; dx\,dy$

$\vec{F} = \langle P, Q \rangle$.

$= \int_{\partial M} \omega = \int_C P\,dx + Q\,dy$. $\quad$ Green's theorem!

3) $M =$ 

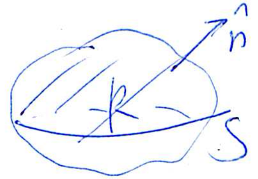
$\quad \partial M = C$. $\quad \subset \mathbb{R}^3$

$\omega = P\,dx + Q\,dy + R\,dz$. $\quad d\omega = (R_y - Q_z)\, dy \wedge dz + (P_z - R_x)\, dz \wedge dx + (Q_x - P_y)\, dx \wedge dy$

Stokes: $\int_M d\omega = \iint_S \cdots$

$= \iint_S \operatorname{curl} \vec{F} \cdot \hat{n}\, dS \qquad \vec{F} = \langle P, Q, R \rangle$.

$= \int_{\partial M} \omega = \int_C P\,dx + Q\,dy + R\,dz$. $\quad$ Stokes' theorem!

4) $M =$ 

$\partial M = S \subset \mathbb{R}^3$

$\omega = P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy \qquad d\omega = (P_x + Q_y + R_z)\,dx\wedge dy\wedge dz$

Stokes: $\int_M d\omega = \iiint_R (P_x + Q_y + R_z)\,dx\wedge dy\wedge dz = \iiint_R \operatorname{div}\vec{F}\,dxdydz$

$= \int_{\partial M} \omega = \iint_S P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy = \iint_S \vec{F}\cdot\hat{n}\,dS$

Exercises

1) $\iint_{\partial R} f\cdot\hat{n}\,dS = \iiint_R \nabla f\cdot dV$

$\Leftrightarrow \int_{\partial R} f\langle dy\wedge dz, dz\wedge dx, dx\wedge dy\rangle = \int_R \langle f_x, f_y, f_z\rangle\,dx\wedge dy\wedge dz$.

2) $\iint_{\partial R} \vec{F}\times\hat{n}\,dS = -\iiint_R \nabla\times\vec{F}\,dV$

$\Leftrightarrow \int_{\partial R} \langle P,Q,R\rangle\times\langle dy\wedge dz, dz\wedge dx, dx\wedge dy\rangle = -\int_R \langle R_y - Q_z, P_z - Q_x, Q_x - P_y\rangle\,dx\wedge dy\wedge dz$

3) $\int_{\partial S} f\cdot d\vec{r} = -\iint_S \nabla f\times\hat{n}\,dS$.

$\Leftrightarrow \int_{\partial S} f\langle dx, dy, dz\rangle = -\int_S \langle f_x, f_y, f_z\rangle\times\langle dy\wedge dz, dz\wedge dx, dx\wedge dy\rangle$

Corollary. M k-dim, $\omega\in\Omega^k$.

Then $\int_M \omega = 0$,

if. • $\omega = d\eta$ exact and $\partial M = \phi$.

or • $d\omega = 0$ and $M = \partial N$. for some $(k+1)$-dim N.
$\omega\in\Omega^k(N)$, $d\omega\in\Omega^{k+1}(N)$.

Pf: $\int_M \overset{=d\eta}{\omega} = \int_{\partial M}\eta = 0$.

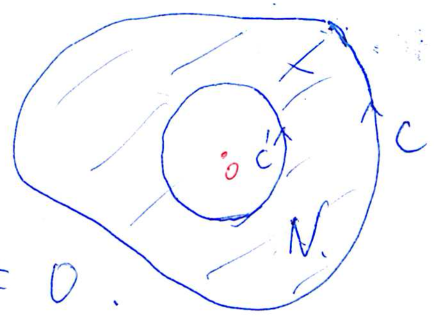
or $\int_{\underset{=\partial N}{M}} \omega = \int_N d\omega = 0$.

Example: • $\vec{F}$ gradient field $= \nabla f$, defined on $\mathbb{R}^2 - \{0\}$

$\oint_C \vec{F}\cdot d\vec{r} = 0$.

• $\operatorname{curl}\vec{F} = 0$, defined on N

$\int_C \vec{F}\cdot d\vec{r} - \int_{C'} \vec{F}\cdot d\vec{r} = \int_{\underset{=\partial N}{C-C'}} \vec{F}\cdot d\vec{r} = 0$.



• However, $\operatorname{curl}\vec{F} = 0$, $\vec{F}$ defined on $\mathbb{R}^2 - \{0\}$.

$\oint_C \vec{F}\cdot d\vec{r}$ may not be 0.

Pf of Stokes' theorem. $\int_{\partial M}\omega = \int_M d\omega$.

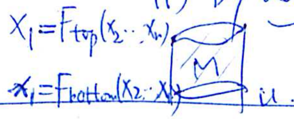
- Case: $M \subset \mathbb{R}^n$, $\dim M = n$, $\omega \in \Omega^{n-1}(M)$.

"divergence thm" $\omega = \sum_{i=1}^{n} f_i \, dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n$.

Simplification i) By linearity, assume $\omega = f_1\, dx_2\wedge\cdots\wedge dx_n$

enough to show $\int_{\partial M} f_1\, dx_2\wedge\cdots\wedge dx_n = \int_M \frac{\partial f_1}{\partial x_1} dx_1\wedge\cdots\wedge dx_n$

ii) By additivity, assume M vertically simple.



Main part: $\int_{\partial M} f_1\, dx_2\wedge\cdots\wedge dx_n = \int_M \frac{\partial f_1}{\partial x_1} dx_1\wedge\cdots\wedge dx_n$.

$$\int_{\partial M} = \int_{top} + \int_{bottom} + \int_{side}.$$

$\varphi: U \longrightarrow top$

$$\int_{top} f_1\, dx_2\wedge\cdots\wedge dx_n = \int_U \varphi^* f_1 (dx_2\wedge\cdots\wedge dx_n)$$
$$= \int_U f_1(F_{top}(x_2,\cdots,x_n), x_2,\cdots,x_n)\, dx_2\wedge\cdots\wedge dx_n$$
$$= \int\!\!\int\cdots\int_U f_1(\cdots)\, dx_2\cdots dx_n$$

$$\int_M \frac{\partial f_1}{\partial x_1} dx_1\wedge\cdots\wedge dx_n \overset{\text{definition}}{=} \int\!\!\int\cdots\int_M \frac{\partial f}{\partial x_1} dx_1\cdots dx_n$$
$$\overset{\text{Fubini}}{=} \int\cdots\int\left(\int_{F_{bottom}}^{F_{top}} \frac{\partial f}{\partial x_1} dx_1\right) dx_2\cdots dx_n$$
$$= \int\cdots\int_U \left(f(F_{top}(x_2,\cdots,x_n), x_2,\cdots,x_n) - f(F_{bottom}(x_2,\cdots,x_n), x_2,\cdots,x_n)\right) dx_2\cdots dx_n$$

$\square$

- General Case. $M^k \subset \mathbb{R}^n$. $\dim M = k$. $\omega \in \Omega^{k-1}(M)$.

$$\int_{\partial M}\omega = \int_M d\omega.$$

Suppose $\varphi: U \subset \mathbb{R}^k \longrightarrow M \subset \mathbb{R}^n$ "parametrization", $d\omega \in \Omega^k(M)$

Then $\int_M d\omega := \int_U \varphi^*(d\omega) \overset{(*)}{=} \int_U d(\varphi^*\omega)$

$\overset{\text{Stokes}}{=} \int_{\partial U}\varphi^*\omega =: \int_{\partial M}\omega$

$\square$

Lemma. $d\varphi^* = \varphi^* d$

Pf: - Check directly that $\varphi^* df = d\varphi^* f = d(f\circ\varphi)$ "Chain rule"

- Suppose $\omega = f\, dy_1\wedge\cdots\wedge dy_k \in \Omega^k$.

$$\varphi^* d\omega = \varphi^*(df\wedge dy_1\wedge\cdots\wedge dy_k) = d(f\circ\varphi)\wedge d(y_1\circ\varphi)\wedge\cdots\wedge d(y_k\circ\varphi).$$

$$d\varphi^*\omega = d(f\circ\varphi\, d(y_1\circ\varphi)\wedge\cdots\wedge d(y_k\circ\varphi))$$
$$= d(f\circ\varphi)\wedge d(y_1\circ\varphi)\wedge\cdots\wedge d(y_k\circ\varphi).$$

"Duality" between M and ω.

M: k-dim mfld, ω: k-form.

<u>Additivity</u>: $\int_{M_1+M_2} \omega = \int_{M_1} \omega + \int_{M_2} \omega$.

<u>Linearity</u>: $\int_M \omega_1 + \omega_2 = \int_M \omega_1 + \int_M \omega_2$.

$d^2\omega = 0$.

$\partial^2 M = \phi$.

$\omega \in \Omega^k(X)$: $d\omega = 0$ "closed"

$\omega = d\eta$ "exact".

exact $\Longrightarrow$ closed, closed $\not\Longrightarrow$ exact

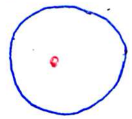
$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \in \Omega^1(\mathbb{R}^2 - \{0\}).$$

$M \subset X$, $\partial M = \phi$ "closed"

$M = \partial N$ "boundary" for some $N \subset X$

boundary $\Longrightarrow$ closed, closed $\not\Longrightarrow$ boundary

$X = \mathbb{R}^2 - \{0\}$

$M =$ 

<u>De Rham cohomology</u>, X^n

$$0 \to \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \to \cdots \to \Omega^n(X) \xrightarrow{d} 0.$$

Let $Z^k(X) = \{\text{closed } k\text{-forms}\} \subset \Omega^k(X)$.

$= \{\ker: \Omega^k(X) \xrightarrow{d} \Omega^{k+1}(X)\}$

$B^k(X) = \{\text{exact } k\text{-forms}\} \overset{d^2=0}{\subset} Z^k(X)$.

$= \{\mathrm{Im}: \Omega^{k-1} \xrightarrow{d} \Omega^k(X)\}$.

Define $H^k(X) = \dfrac{Z^k(X)}{B^k(X)}$ k^{th} <u>cohomology gp</u>.

<u>Nash's problem</u>: Find a subset X of $\mathbb{R}^3$ s.t.

"A beatiful Mind" in

$V := \{\vec{F}: \mathbb{R}^3 - X \to \mathbb{R}^3 \text{ so } \nabla \times \vec{F} = 0\}$

$W := \{\vec{F} = \nabla g\}$.

then: $\dim(V/W) = 8$.

"As I said, this problem will take some of you a few months to solve, for others, the rest of your lives."

This is equivalent to $H^1(\mathbb{R}^3 - X) = \mathbb{R}^8$.

not really a calculus problem.

X simply connected $\Longrightarrow H^1(X) = 0$.