1. For the derivative of $(f\cdot g)$, the definition formula is:

$$(f\cdot g)'(c) = \left.\frac{d(f\cdot g)}{dx}\right|_{x=c} = \lim_{h\to 0}\frac{f(c+h)\,g(c+h) - f(c)\cdot g(c)}{h}$$

$$= \lim_{h\to 0}\Big[\underbrace{\frac{f(c+h)\,g(c+h) - f(c+h)\cdot g(c)}{h}}_{\mathrm{I}} + \underbrace{\frac{f(c+h)\cdot g(c) - f(c)g(c)}{h}}_{\mathrm{II}.}\Big] \text{ (by hint, insert } f(c+h)g(c))$$

consider (II) first:

$$\lim_{h\to 0}\frac{f(c+h)\,g(c) - f(c)g(c)}{h} = g(c)\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} \quad (g(c) \text{ is independent with } h)$$

$$\Downarrow$$

$$= g(c)\cdot f'(c) \quad \text{(definition of } f'(c)\text{, and we know } f(x) \text{ is differentiable at } x=c)$$

then for (I), First we try to show $\lim_{h\to 0} f(c+h) = f(c)$, actually this is continuity at $x=c$.

$$\lim_{h\to 0}[f(c+h) - f(c)] = \lim_{h\to 0}\left[\frac{f(c+h)-f(c)}{h}\cdot h\right] \overset{(*)}{=} \underbrace{\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}}_{(1)}\cdot \underbrace{\lim_{h\to 0} h}_{(2)} = f'(c)\cdot 0 = 0 \Rightarrow \lim_{h\to 0} f(c+h) = f(h)$$

(*) is true for (1) (2) both exist, then we can have $\lim (1)\cdot(2) = \lim(1)\lim(2)$.

this shows differentiable $\Rightarrow$ continuity.

so back (I):

$$\lim_{h\to 0}\frac{f(c+h)\,g(c+h) - f(c+h)\cdot g(c)}{h} = \lim_{h\to 0} f(c+h)\cdot\frac{g(c+h)-g(c)}{h} = \lim_{h\to 0} f(c+h)\cdot\lim_{h\to 0}\frac{g(c+h)-g(c)}{h} = f(c)\cdot g'(c).$$

combining all above, we have:

$$(f\cdot g)'(c) = \lim_{h\to 0}(\mathrm{I}) + \lim_{h\to 0}(\mathrm{II}) = f(c)g'(c) + g(c)f'(c)$$

$$\Downarrow$$

(for $\lim(\mathrm{I})$, $\lim(\mathrm{II})$ both exist)

2. (a) $f(x) = g\left(\frac{x}{1+g^2(x)}\right)$

regard $u = \frac{x}{1+g^2(x)}$ as intermediate variable

So $f'(x) = \frac{dg}{du}\cdot\frac{du}{dx} = g'(u)\cdot u'(x)$ (chain-rule)

while $u'(x) = \frac{(x)'(1+g^2(x)) - x\cdot[1+g^2(x)]'}{(1+g^2(x))^2}$ (Quotient rule)

$$= \frac{1+g^2(x) - x\cdot 2g(x)\cdot g'(x)}{[1+g^2(x)]^2}$$

$$\Rightarrow f'(x) = g'\left(\frac{x}{1+g^2(x)}\right) \cdot \frac{1+g^2(x) - 2x \cdot g \cdot g'(x)}{(1+g^2(x))^2}$$

(b). $k(x) = e^{xg(x)}$ $\quad u = xg(x)$

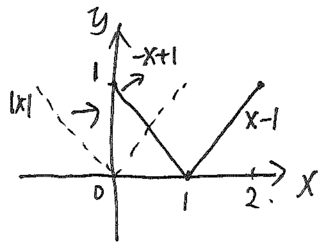
$$k'(x) = (e^u)' \cdot u'(x) = e^u \cdot [g(x) + x \cdot g'(x)] = e^{xg(x)}[g + xg'] \text{ (chair-rule)}$$

$$k''(x) = [\underbrace{e^{xg(x)}}_{k(x)}]' [g+xg'] + e^{xg(x)}[g+xg']' \text{ (product-rule)}$$

$$= \underbrace{e^{xg(x)} [g+xg']^2}_{\text{(from } k'(x))} + e^{xg(x)}[g' + g' + xg'']$$

$$= e^{xg(x)}[(g+xg')^2 + 2g' + xg'']$$

3. A very simple example is to translate the $|x|$ like:



$$f(x) = \begin{cases} -x+1 & 0 \le x \le 1 \\ x-1 & 0 < x \le 2 \end{cases}$$

$f(x)$ is non-differentiable at $x=1$

4. (a) we apply $e^{\ln f(x)} = f(x)$ here:

$$\lim_{x\to 0^+} x^{\sin x} = \lim_{x\to 0^+} e^{\sin x \cdot \ln x} = e^{\lim_{x\to 0^+} \sin x \cdot \ln x} \text{ (due to the continuity of } e^x)$$

$$\text{And } \lim_{x\to 0^+} \sin x \cdot \ln x = \lim_{x\to 0^+} \frac{\sin x}{x} \cdot x \ln x = \lim_{x\to 0^+} \frac{\sin x}{x} \cdot \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \rightarrow \text{(L'Hospital rule.}$$

$$= 1 \cdot \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 1 \cdot \lim_{x\to 0^+} (-x) = 0$$

$$\Rightarrow \lim_{x\to 0^+} x^{\sin x} = e^0 = 1$$

(b) $\displaystyle \lim_{x\to 0} (\cot x - \frac{1}{x}) = \lim_{x\to 0} \frac{x \cdot \cos x - \sin x}{x \sin x}$ (L'Hospital rule)

$$= \lim_{x\to 0} \frac{\cos x - x \sin x - \cos x}{\sin x + x \cdot \cos x}$$

$$= -\lim_{x\to 0} \frac{x \sin x}{\sin x + x \cdot \cos x} = -\lim_{x\to 0} \frac{\sin x}{\frac{\sin x}{x} + \cos x}$$

$$= -\frac{\lim_{x\to 0} \sin x}{\lim_{x\to 0}(\frac{\sin x}{x} + \cos x)} = -\frac{0}{1} = 0.$$