Application 1

Thm 3 Let
- M = complete simply-connected Riem. mfd with
- $K \leq 0$ (sectional curvature)
- $O \in M$ is a fixed point.
- $\rho : M \to [0, \infty)$ (the distance function wrt O) is defined by

$$\rho(x) = d(x, O).$$

Then $\rho^2 \in C^\infty(M)$ and $D^2 \rho^2 > 0$

(strictly positive definite)

Eg: If $M = \mathbb{R}^n$, $O =$ origin, then $\rho^2(x) = |x|^2$ &

$$D^2\rho^2(v,v) = c|v|^2 \quad (\text{for some } c > 0)$$

Pf of Thm3: By Cartan-Hadamard Thm,

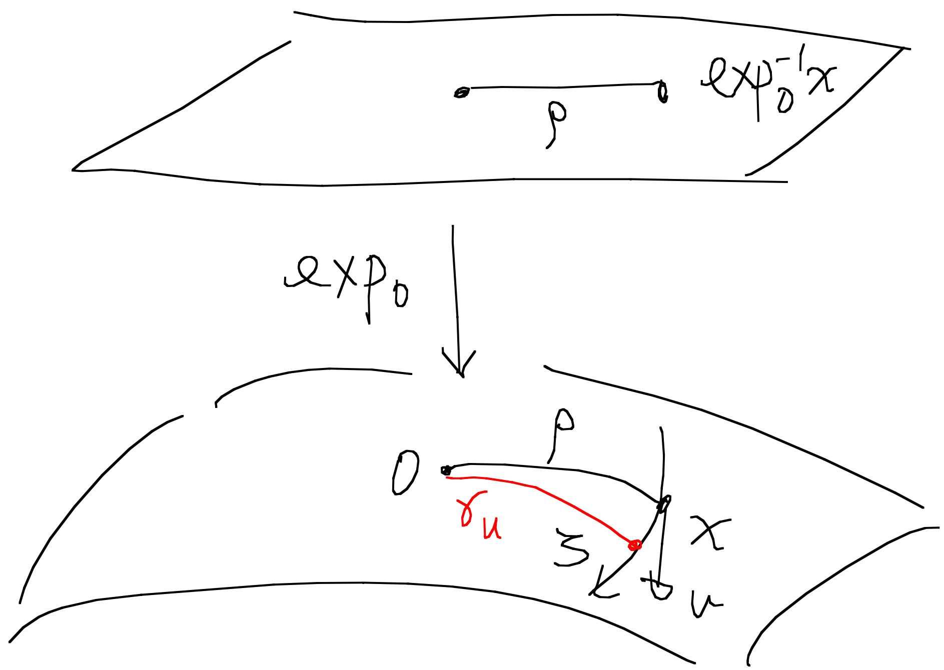
$$\rho(x) = |(\exp_O)^{-1}(x)|,$$

Therefore $\rho^2(x) = |(\exp_O)^{-1}(x)|^2$ is smooth.

Now suppose $x \neq O$, and $v \in T_xM$

Take a curve $\zeta : [-\varepsilon, \varepsilon] \to M$ s.t.

$$\zeta(0) = x \quad \& \quad \zeta'(0) = v.$$



For each $u \in [-\varepsilon, \varepsilon]$, let $\gamma_u : [0, b] \to M$ (with $b = \rho(x)$) is the unique geodesic joining O to $\zeta(u)$. Note that $\gamma_0 = \gamma : [0, b] \to M$ is a normalized geodesic (other γ_u may not be a normalized geodesic).

In particular, we can choose $\zeta(u)$ to be a geodesic.

Then, the end point of γ_u is $\gamma_u(b) = \zeta(u)$

$\Rightarrow$ the transversal vector field $V(t,u)$ at $t=b$

is $V(b,u) = \zeta'(u)$.

Therefore $D_V V\big|_{(b,u)} = D_{\zeta'(u)}\zeta'(u) = 0$ since ζ = geodesic.

Also, at $t=0$, we have $\gamma_u(0) = 0$ & $V(0,u) \equiv 0$

$\Rightarrow$ $D_V V\big|_{(0,u)} = 0$.

Hence, the 2^{nd} variation formula gives

$$\frac{d^2 L}{du^2}(0) = \int_a^b \{ |D_{\gamma'} V^{\perp}|^2 - \langle R_{V^{\perp}\gamma'} V^{\perp}, \gamma' \rangle \} \, dt$$

$$\geq \int_a^b |D_{\gamma'} V^{\perp}|^2 \qquad (\text{since } K \leq 0).$$

Now $D^2 \rho^2 (v,v) = \{ \zeta'(\zeta' \rho^2) - (D_{\zeta'} \zeta') \rho^2 \} \Big|_{u=0}$

$$= \zeta'(\zeta' \rho^2) \Big|_{u=0} \qquad (\text{since } \zeta = \text{geodesic})$$

$$= \zeta'(2\rho \, \zeta' \rho) \Big|_{u=0}$$

$$= \left[2\rho \, \zeta'(\zeta' \rho) + 2(\zeta' \rho)^2 \right] \Big|_{u=0}$$

$$= 2\rho(x) \frac{d^2}{du^2}\Big|_{u=0} \rho(\zeta(u)) + 2\left[\frac{d}{du}\Big|_{u=0} \rho(\zeta(u)) \right]^2$$

Note that $\rho(\zeta(u)) = L(\gamma_u) = L(u)$

$$\therefore \quad \frac{d}{du}\Big|_{u=0} \rho(\zeta(u)) = \frac{dL}{du}(0)$$

$$= \langle \gamma'(t), U(t) \rangle \Big|_0^b - \int_0^b \langle D_{\gamma'} \gamma', U \rangle \, dt$$

$$= + \langle \gamma'(b), U(b) \rangle \qquad \left(\begin{array}{c} \text{since } \gamma = \text{geodesic} \\ \& \ U(0) = 0 \end{array} \right)$$

$$= + \langle \gamma'(b), \zeta'(0) \rangle$$

$$= + \langle \gamma'(b), v \rangle$$

$$\& \quad \frac{d^2}{du^2}\Big|_{u=0} \rho(\zeta(u)) = \frac{d^2 L}{du^2}(0) \geq \int_0^b |D_{\gamma'} U^\perp|^2 \, dt$$

$$\therefore \quad D^2 \rho^2 (v, v) \geq 2\rho(x) \int_0^b |D_{\gamma'} U^\perp|^2 dt + 2\left[\langle \gamma'(b), v \rangle\right]^2$$

If $\langle \gamma'(b), v \rangle \neq 0$, then $D^2 \rho^2 (v, v) > 0$

If $\langle \gamma'(b), v \rangle = 0$, then $U(b) = v \in [\gamma'(b)]^{\perp}$

Note that $\{\gamma_u\}$ is a 1-para. family of geodesics, U is a Jacobi field along γ. Hence

$$\langle \gamma'(b), U(b) \rangle = \langle \gamma'(0), U(0) \rangle = 0$$

$$\& \quad U(b) = v \neq 0$$

$\Rightarrow U(t)$ is a <u>nontrivial</u> normal Jacobi field

$\therefore \quad U^{\perp}(t) = U(t)$.

Therefore $D_{\gamma'} U^{\perp} = D_{\gamma'} U \neq 0$. Otherwise, U is a parallel transport of $V(0)=0 \Rightarrow V \equiv 0$ which is a contradiction. Hence

$$D^2\rho^2(v,v) \geq \int_0^b |D_{\gamma'}V^{\perp}|^2 dt > 0.$$

This completes the proof of the thm. #

The <u>key</u> point of the conclusion of the above thm is that $D^2\rho^2 > 0$ on the <u>whole</u> M, which needs the curvature assumption. Otherwise, we have

<u>Lemma 4</u> Let • M = Riem mfd

- $O \in M$
- $\rho : M \to \mathbb{R}$ distance to O.

Then $\exists$ a nbd $\mathcal{U}_0$ of O in M s.t.

ρ^2 is smooth and $D^2\rho^2 > 0$ in $\mathcal{U}_0$.

Pf: Let $\mathcal{U}$ be a nbd of O s.t. $\exists$ normal coordinate system $\{x^1, \cdots, x^n\}$ centered at O. Using this one can show that $\forall\, v, w \in T_O M$,

$$D^2\rho^2(v, w) = 2\langle v, w\rangle \quad (\text{Ex!})$$

Therefore, at the center O, $D^2\rho^2 > 0$.

$\Rightarrow D^2\rho^2 > 0$ in a nbd $\mathcal{U}_0 \subset \mathcal{U}$ of O. #

Recall: • A function $f: M \to \mathbb{R}$ (M = Riem. mfd)

is said to be convex (strictly convex)

$\iff$ $\forall$ geodesic γ in M,

$f \circ \gamma$ is convex (strictly convex)

• Therefore, a C^∞ $f: M \to \mathbb{R}$ is convex (strictly convex) $\iff$ $D^2 f \geq 0$ (> 0) (Ex!)

Def: Let M = complete Riem. mfd. Then

• a subset $\Omega \subset M$ is called convex

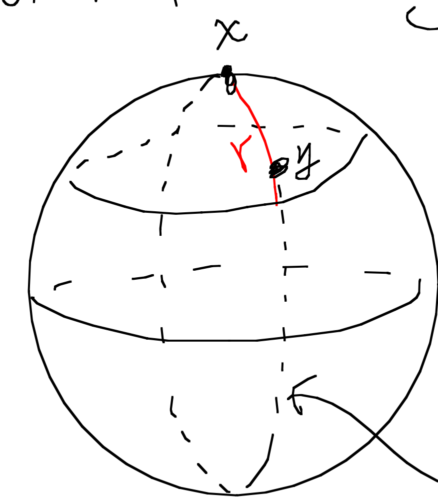
$\iff$ $\forall x, y \in \Omega$, the shortest geodesic joining

x to y is contained in Ω.

- a subset $\Omega \subset M$ is called <u>totally</u> <u>convex</u>

$\iff \forall x, y \in \Omega$, <u>any</u> geodesic joining x to y is contained in Ω.

<u>Eg1</u>: On $S^2 \subset \mathbb{R}^3$, geodesic ball of radius $r \le \frac{\pi}{2}$ is convex, but not totally convex:

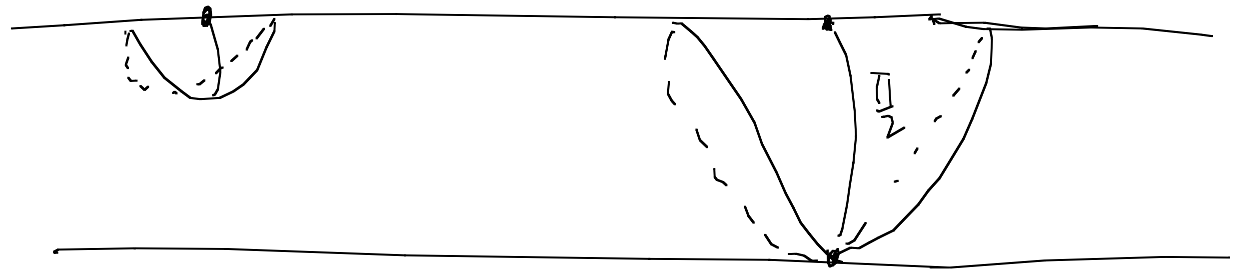


geodsic $\not\subset B_r(x)$ joing x to y.

Furthermore, geodesic ball of radius r between $\frac{\pi}{2}$ & π, is not even convex. (Ex!)

Note: If M is a simply-connected complete Riem. mfd with nonpositive sectional curvature. Then Cartan–Hadamard $\Rightarrow$ any geodesic is minimizing. Therefore, a convex subset of M is also totally convex.

Eg2 Cylinder $\{x^2+y^2=1\} \subset \mathbb{R}^3$. Then B_r is convex for $r \leq \frac{\pi}{2}$, not convex for $r > \frac{\pi}{2}$:



Lemma 5 Let M = Riem mfd

(1) Let $\tau : M \to \mathbb{R}$ is a convex function

- $M_c \overset{\text{def}}{=} \{x \in M : \tau(x) < c\}$ be the sublevel set.

- $\gamma : [a,b] \to M$ be a geodesic.

Then $\gamma(a), \gamma(b) \in M_c \Rightarrow \gamma([a,b]) \subset M_c$.

(2) Furthermore, if M is complete, then M_c is totally convex.

Pf: (1) $\tau\circ\gamma(t) \leq \max\{\tau\circ\gamma(a), \tau\circ\gamma(b)\} < c$

since $\tau\circ\gamma$ is convex,

(2) Easily follows from (1). ※

Cor (of Thm 3): Geodesic balls of a simply-connected complete Riem. mfd M with nonpositive sectional curvature are totally convex.

In particular, $\forall x \in M$, $\{x\}$ is totally convex.

Therefore, there is no nontrivial geodesic $\gamma: [a,b] \to M$ s.t. $\gamma(a) = \gamma(b) = x$.

Thm 6 (J.H.C. Whitehead) Let M = Riem. mfd. Then $\forall x \in M$, $\exists$ a <u>convex</u> nbd. of x.

<u>Pf</u>: $\forall x \in M$, Lemma 4 (& properties of $\exp_x$)

$\Rightarrow \exists \, \varepsilon > 0$ s.t.

- $\exp_x : \underset{\subset T_xM}{B(\varepsilon)} \longrightarrow \underset{\subset M}{B_\varepsilon(x)}$ is a diffeomorphism
- $B_\varepsilon(x) = \exp_x(B(\varepsilon))$ has compact closure in M (note that M may not be complete)
- ρ^2 is C^∞ & $D^2\rho^2 > 0$ on $B_\varepsilon(x)$, where ρ = distance to x.

In fact, by choosing a smaller $\varepsilon > 0$, we can also assume that $\forall\, y \in B_\varepsilon(x)$, $\exp_y\big|_{B(\varepsilon)}$ is a diffeomorphism.

Let $\delta = \frac{\varepsilon}{3} > 0$ and consider the geodesic ball $B_\delta(x)$.

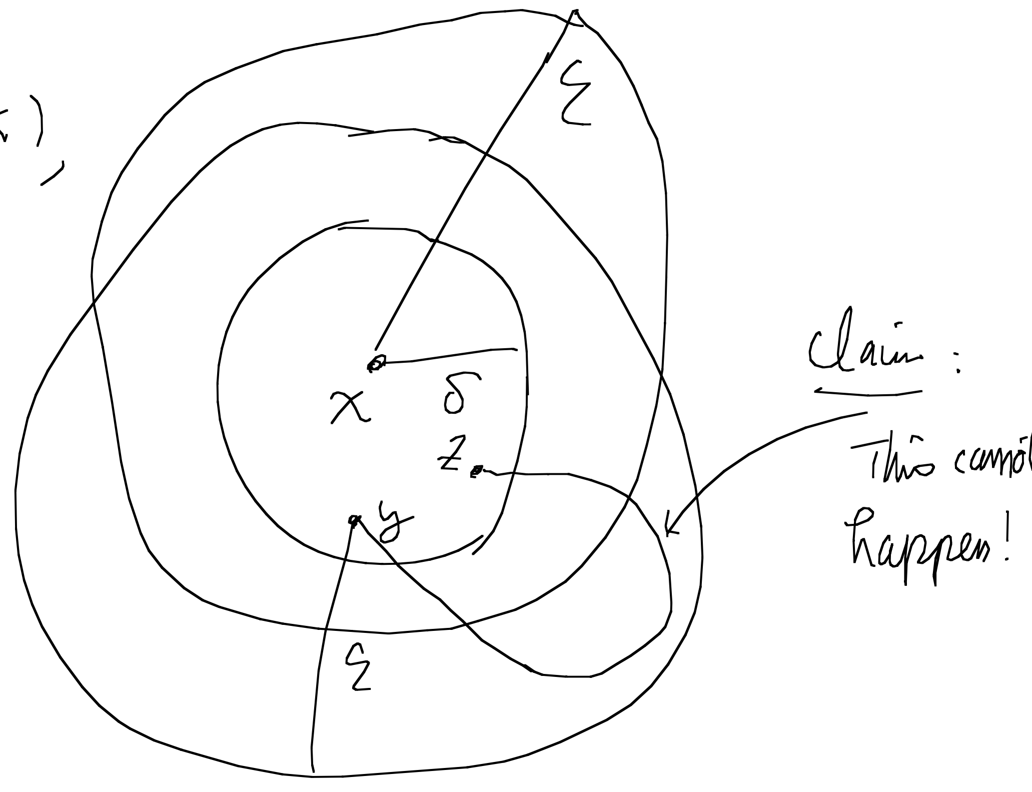
Then for a fixed $y \in B_\delta(x)$,

we claim that

$B_\delta(x) \subset B_\varepsilon(y)$.

In fact, $\forall\, z \in B_\delta(x)$,

$d(z, y) \leq d(z, x) + d(x, y)$



$$\leq \delta + \delta = 2\delta = \frac{2\varepsilon}{3} < \varepsilon .$$

Therefore $\forall z \in B_\delta(x)$, $\exists$ shortest geodesic ζ joining z to y with $\zeta \in B_\varepsilon(y)$ and $L(\zeta) < \varepsilon$.

Claim: $\zeta \subset B_\varepsilon(x)$.

Otherwise, $y, z \in B_\delta(x) \Rightarrow$

$$L(\zeta) > 2(\varepsilon - \delta) = 2\left(\varepsilon - \frac{\varepsilon}{3}\right) = \frac{4}{3}\varepsilon > \varepsilon$$

which is a contradiction.

Since $D^2\rho^2 > 0$ on $B_\varepsilon(x)$, statement (1) of Lemma 5

$\Rightarrow$ the sublevel sets of ρ^2 are convex

$\Rightarrow$ $B_\delta(x)$ is convex

$\Rightarrow$ $\xi \subset B_\delta(x)$ since ξ is the shortest geodesic joining z to y.

Since $y \in B_\delta(x)$ is arbitrary, we've shown that $\forall\, y, z \in B_\delta(x)$, $\exists$ shortest geodesic $\xi \subset B_\delta(x)$ joining y & z, $\therefore$ $B_\delta(x)$ is convex. $\#$

Application 2: <u>Synge Thm</u>

<u>Recall</u>: • A C^∞ mfd M of n-dimension is said to be

orientable $\Leftrightarrow$ $\exists$ a nowhere zero C^∞ n-form ω on M.

- If such an ω is chosen, then it is called the orientation of M. ($\omega_1 \sim \omega_2 \Leftrightarrow \omega_1 = f\omega_2$ for some function $f > 0$)

- Let ω be a nowhere zero n-form on such an M, then bases of T_xM can be divided into 2 classes:

$$\begin{cases} \text{positive oriented} : & \omega(e_1, \cdots, e_n) > 0 \\ \text{negative oriented} : & \omega(e_1, \cdots, e_n) < 0 \end{cases} \quad (\text{wrt } \omega)$$

Lemma 7: Let $\gamma = [a,b] \to M$ be a closed curve in an orientable Riem. mfd M s.t. $x = \gamma(a) = \gamma(b)$.

Then the parallel transport along γ

$$P^\gamma : T_xM \to T_xM \quad \text{has } \det P^\gamma = +1.$$

Pf: (Ex!)

Lemma 8: Let M = non simply-connected compact Riem. mfd. $(\pi_1(M) \neq 1)$

Then $\exists$ closed curve $\gamma : [0,b] \to M$ (for some $b > 0$) such that $L(\gamma) \leq L(\alpha)$ for any piecewise

C^∞ closed curve α which is non-homotopic to zero (ie $\alpha \not\sim 1$).

(Pf: Omitted)

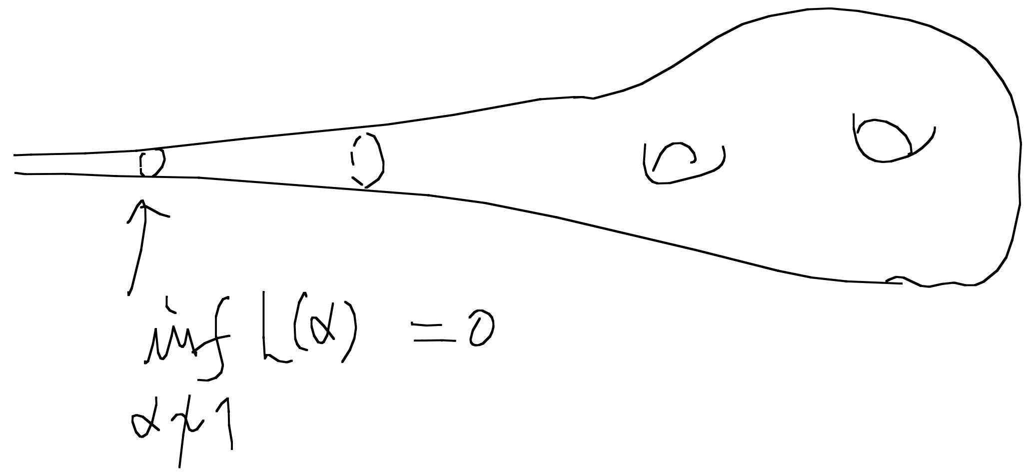
Note: • $\pi_1(M) \neq 1$ is necessary: any closed curve can be shrinked to a point

$$\Rightarrow \inf_\gamma L(\gamma) = 0$$

$\Rightarrow$ no curve minimizing the length functional.

• <u>compactness</u> is also necessary:

eg: surface with cusp:



Thm 9 (J.L. Synge) If M is a compact orientable even dimensional Riem. mfd with positive sectional curvature, then M is simply-connected.

Pf: Suppose not, then $\pi_1(M) \neq 1$.

By Lemma 8, $\exists$ a closed curve $\gamma : [0,b] \to M$ s.t. $L(\gamma) \leq L(\alpha)$, $\forall\, \alpha \neq 1$.

Then γ has to be a geodesic and hence

$$\gamma'(0) = \gamma'(b).$$

We may also assume $|\gamma'(t)| = 1$,

Let $x = \gamma(0) = \gamma(b)$. Then parallel transport along γ :

$$P^\gamma = T_x M \to T_x M$$

has $\det P^\gamma = +1$.

Note that eigenvalues of P^γ are of the form ± 1, $e^{i\theta}$ $(\neq \pm 1)$, and if $e^{i\theta}$ is an eigenvalue, then $e^{-i\theta}$ is also an eigenvalue.

Since $\det P^\gamma = +1$, the dim{-1-eigenspace} is even. Hence $\dim M = \text{even} \Rightarrow$

dim{$+1$-eigenspace} is also even.

Note that γ is a closed geodesic

$$\gamma'(0) = \gamma'(b) \quad \& \quad P^\gamma(\gamma'(0)) = \gamma'(b) = \gamma'(0)$$

$\Rightarrow$ dim{$+1$ eigenspace} > 0, hence ≥ 2.

Therefore, $\exists\, e \in T_xM$ s.t. $P^\gamma(e) = e$,

$$\langle e, \gamma'(0) \rangle = 0$$

Now, let V be a parallel vector field along γ

s.t. $V(0)=e$.

Then $V(b)=P^{\gamma}(V(0))=P^{\gamma}(e)=e$

$\therefore$ V is a well-defined vector field on the closed curve γ.

$\Rightarrow$ $\exists$ a 1-parameter family of closed geodesics $\{\gamma_u\}$ s.t. $\gamma_0=\gamma$ & $V=$ transversal vector field of $\{\gamma_u\}$. $\left(\gamma_u(t)=\exp_{\gamma(t)}(uV(t)),\ |u|\ll 1\right)$

Then, 2^{nd} variation formula

$$\Rightarrow \frac{d^2L}{du^2}(0)=\int_0^b\left[|D_{\gamma'}V^{\perp}|^2-\langle R_{\gamma' V^{\perp}}\gamma', V^{\perp}\rangle\right]dt$$

(γ_u closed $\forall u \Rightarrow$ bdy term $= 0$)

Since $\langle V(0), \gamma'(0)\rangle = \langle e, \gamma'(0)\rangle = 0$, we have

$$V^{\perp} = V \quad , \quad \forall t \in [0,b].$$

$\Rightarrow \quad D_{\gamma'} V^{\perp} = D_{\gamma'} V = 0$ (since V is parallel)

$$\Rightarrow \quad \frac{d^2 L}{du^2}(0) = - \int_0^b \langle R_{\gamma' V^{\perp}} \gamma', V^{\perp} \rangle \, dt$$

< 0 since sectional curvature > 0.

Contradicting the assumption that γ is length minimizing. $\therefore \pi_1(M) = 1$ #