Lemma 6 $\left(B^n, \dfrac{4}{(1-|x|^2)^2} \sum_{i=1}^{n} dx^i \otimes dx^i\right)$ where $|x|^2 = \sum_{i=1}^{n} (x^i)^2$.

is a complete Riemannian manifold with constant sectional curvature -1.

Pf: (1) Completeness

Pf: First note that $\forall A \in O(n)$

$$A|_{B^n} : B^n \to B^n \text{ is an isometry}$$

of the hyperbolic geometry

(A preserves $|x|$ & $\sum dx^i \otimes dx^i$)

Now consider the curve

$$\xi(s) : (-\infty, \infty) \longrightarrow \mathbb{B}^n$$

$$\rotatebox{90}{$\in$}$$

$$s \longmapsto \left(\frac{e^s - 1}{e^s + 1}, 0, \cdots, 0 \right)$$

Then $\xi'(s) = \left(\frac{2e^s}{(e^s+1)^2}, 0, \cdots, 0 \right)$

$$\Longrightarrow \quad |\xi'(s)|^2_{hyp} = \frac{4}{(1-|\xi|^2)^2} \left[\frac{2e^s}{(e^s+1)^2} \right]^2 \overset{(ex)}{=} 1$$

$\therefore$ ξ is arc-length parametrized.

Let $A \in O(n)$ be given by

$$A(x^1, x^2, \cdots, x^n) = (x^1, -x^2, \cdots, -x^n).$$

Then $\xi((-\infty, \infty)) = \{ x \in \mathbb{B}^n : Ax = x \}$

Lemma 4 $\Rightarrow$ ξ is a normalized geodesic defined on the whole $(-\infty, \infty)$ with $\xi'(0)$ in the e_1-direction ($\{e_i\}$ = standard basis of $\mathbb{R}^n$)

Applying other $A \in O(n)$, we have geodesic with

$(A\xi)'(0)$ = any given direction

defined on the whole $(-\infty, \infty)$

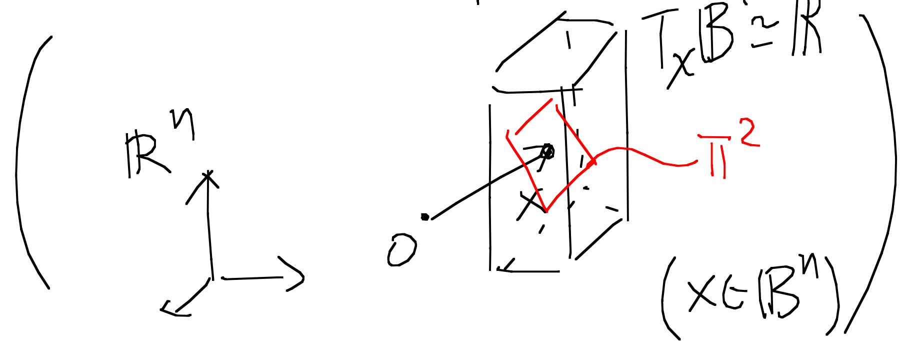
Therefore $\exp_0$ is defined on the whole $T_0\mathbb{B}^n$.

Hence Hopf-Rinow $\Rightarrow$ $\mathbb{B}^n$ is complete. #

(2) <u>Curvature $\equiv -1$</u>.

Pf: Let $x \in B^n$ and $\Pi \subset T_x B^n$ be a 2-plane.

Identify $T_x B^n \cong \mathbb{R}^n$, and x can be considered



as an element in $\mathbb{R}^n$ ($B^n \subset \mathbb{R}^n$).

Assume $n \geq 3$.

Take a 3-dim'l subspace $E \subset \mathbb{R}^n$ s.t

$$\operatorname{span}\{x, \Pi\} \subset E$$

(If $x \neq 0$ & $x \notin \Pi$, then E is unique, otherwise not)

Then $\mathbb{R}^n = E \oplus E^\perp$ orthogonal (in Euclidean)

and one can defines a map

$$\phi : (e, e') \mapsto (e, -e') \qquad e \in E, \ e' \in E^\perp$$

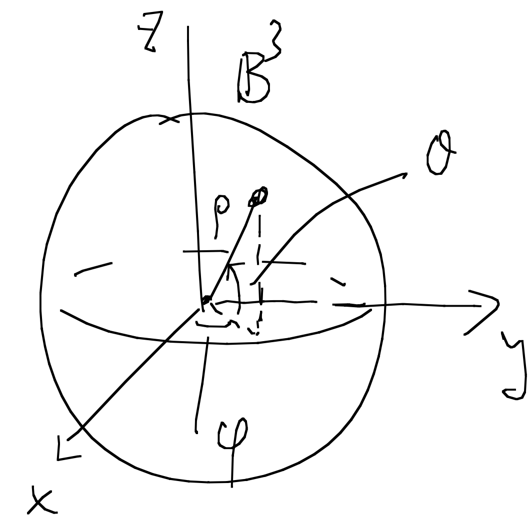
Then $\phi|_{\mathbb{B}^n}$ is an isometry of $\mathbb{B}^n$ with fixed point set $E \cap \mathbb{B}^n$.

$\Rightarrow$ $\mathbb{B}^3 = E \cap \mathbb{B}^n$ is a totally geodesic submanifold of $\mathbb{B}^n$

$\Rightarrow$ $K_{\mathbb{B}^n}(\Pi) = K_{\mathbb{B}^3}(\Pi)$

So we only need to show the case that $n=3$.

Let $\{\rho, \varphi, \theta\}$ = polar coordinates on $\mathbb{B}^3$



$\Rightarrow$ on $\mathbb{B}^3 \setminus \{0\}$, the metric

$$\frac{4}{(1-|x|^2)^2} \sum dx^i \otimes dx^i$$ can be

written as

$$\frac{4}{(1-\rho^2)^2} \left(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \cos^2\theta \, d\varphi^2 \right)$$

where $d\rho^2 = d\rho \otimes d\rho$, ...

Let
$$\begin{cases} e_1 = \dfrac{1-\rho^2}{2}\dfrac{\partial}{\partial \rho} \\ e_2 = \dfrac{1-\rho^2}{2\rho}\dfrac{\partial}{\partial \theta} \\ e_3 = \dfrac{1-\rho^2}{2\rho\cos\theta}\dfrac{\partial}{\partial \varphi} \end{cases}$$

Then $\langle e_i, e_j \rangle = \delta_{ij}$ (Ex.)

$$\Rightarrow \quad \langle D_{e_i} e_j, e_k \rangle = \frac{1}{2}\left\{ \langle e_k, [e_i, e_j] \rangle + \langle e_j, [e_k, e_i] \rangle - \langle e_i, [e_j, e_k] \rangle \right\}$$

Now
$$[e_1, e_2] = \frac{1-\rho^2}{2}\frac{\partial}{\partial \rho}\left(\frac{1-\rho^2}{2\rho}\frac{\partial}{\partial \theta}\right) - \frac{1-\rho^2}{2\rho}\frac{\partial}{\partial \theta}\left(\frac{1-\rho^2}{\rho}\frac{\partial}{\partial \rho}\right)$$
$$= \frac{1-\rho^2}{2}\left(\frac{1-\rho^2}{2\rho}\right)'\frac{\partial}{\partial \theta} = -\frac{1+\rho^2}{2\rho} e_2 \quad \text{(Ex)}$$

Similarly $[e_2, e_3] = \frac{1-\rho^2}{2\rho}\frac{\partial}{\partial\theta}\left(\frac{1-\rho^2}{2\rho\cos\theta}\frac{\partial}{\partial\varphi}\right) - \frac{1-\rho^2}{2\rho\cos\theta}\frac{\partial}{\partial\varphi}\left(\frac{1-\rho^2}{2\rho}\frac{\partial}{\partial\theta}\right)$

$$\overset{(ex)}{=\!=} \frac{1-\rho^2}{2\rho}\tan\theta\, e_3$$

$$[e_1, e_3] = -\frac{1+\rho^2}{2\rho} e_3 \qquad (Ex.)$$

In conclusion

$$\begin{cases} [e_1, e_2] = -\frac{1+\rho^2}{2\rho} e_2 \\ [e_2, e_3] = \frac{1-\rho^2}{2\rho}\tan\theta\, e_3 \\ [e_1, e_3] = -\frac{1+\rho^2}{2\rho} e_3 \end{cases}$$

Then straight faward calculation (Ex) $\Longrightarrow$

$$
\begin{cases}
D_{e_1} e_1 = 0, & D_{e_2} e_1 = \dfrac{1+\rho^2}{2\rho} e_2, & D_{e_3} e_1 = \dfrac{1+\rho^2}{2\rho} e_3 \\
D_{e_1} e_2 = 0, & D_{e_2} e_2 = -\dfrac{1+\rho^2}{2\rho} e_1, & D_{e_3} e_2 = -\dfrac{1-\rho^2}{2\rho} \tan\theta \, e_3 \\
D_{e_1} e_3 = 0, & D_{e_2} e_3 = 0, & D_{e_3} e_3 = -\dfrac{1+\rho^2}{2\rho} e_1 + \dfrac{1-\rho^2}{2\rho} \tan\theta \, e_2
\end{cases}
$$

Hence

$$
\begin{aligned}
R(e_1, e_2, e_1, e_2) &= \langle R_{e_1 e_2} e_1, e_2 \rangle \\
&= \langle D_{[e_1, e_2]} e_1 - [D_{e_1}, D_{e_2}] e_1 , e_2 \rangle \\
&= -\frac{1+\rho^2}{2\rho} \langle D_{e_2} e_1, e_2 \rangle - \langle D_{e_1}(D_{e_2} e_1) - D_{e_2}(\cancel{D_{e_1}} e_2), e_2 \rangle
\end{aligned}
$$

$$= -\left(\frac{1+\rho^2}{2\rho}\right)^2 \langle e_2, e_2\rangle - \left\langle D_{e_1}\left(\frac{1+\rho^2}{2\rho} e_2\right), e_2\right\rangle$$

$$= -\left(\frac{1+\rho^2}{2\rho}\right)^2 - e_1\left(\frac{1+\rho^2}{2\rho}\right)\langle e_2, e_2\rangle - \frac{1+\rho^2}{2\rho}\langle \cancel{D_{e_1}} e_2, e_2\rangle$$

$$= -\left(\frac{1+\rho^2}{2\rho}\right)^2 - \frac{1-\rho^2}{2}\frac{\partial}{\partial\rho}\left(\frac{1+\rho^2}{2\rho}\right)$$

$$= -1 \quad (\text{Ex.})$$

Similarly $R(e_1, e_3, e_1, e_3) = R(e_2, e_3, e_2, e_3) = -1$ (Ex.)

To complete the proof, we need to show that all other

$$R(e_i, e_j, e_k, e_l) = 0. \qquad (\text{Ex})$$

Since $n=3$, the indices have to be repeated.

It is clear that if $i=j=k=l$ or 3 of the indices are equal, then $R(e_i, e_j, e_k, e_l) = 0$.

Therefore, we only need to consider

$$R(e_i, e_j, e_i, e_k) \qquad \text{with } j<k. \ (i,j,k \text{ distinct})$$

Other cases are clear zero or can be reduced to this case. (If $j=k$, it is the previous situation)

For $i=3$,

$$R(e_3, e_1, e_3, e_2) = \langle R_{e_3 e_1} e_3, e_2 \rangle$$

$$= \langle D_{[e_3,e_1]} e_3, e_2 \rangle - \langle D_{e_3} D_{\cancel{e_1}} e_3, e_2 \rangle + \langle D_{e_1} D_{e_3} e_3, e_2 \rangle$$

$$= \frac{1+\rho^2}{2\rho} \langle D_{e_3} e_3, e_2 \rangle + \langle D_{e_1} (D_{e_3} e_3), e_2 \rangle$$

$$= \frac{1+\rho^2}{2\rho}\left\langle -\frac{1+\rho^2}{2\rho} e_1 + \frac{1-\rho^2}{2\rho}\tan\theta\, e_2 , e_2 \right\rangle$$

$$+ \left\langle D_{e_1}\left(-\frac{1+\rho^2}{2\rho} e_1 + \frac{1-\rho^2}{2\rho}\tan\theta\, e_2\right), e_2 \right\rangle$$

$$= \frac{(1+\rho^2)(1-\rho^2)}{4\rho^2}\tan\theta + e_1\left(\frac{1-\rho^2}{2\rho}\tan\theta\right)$$

$$= \frac{1-\rho^4}{4\rho^2}\tan\theta + \frac{1-\rho^2}{2}\left(\frac{1-\rho^2}{2\rho}\right)'\tan\theta$$

$$= 0 \qquad (\text{Ex})$$

Similarly, $R(e_1, e_2, e_1, e_3) = R(e_2, e_1, e_2, e_3) = 0$.

Hence $\mathbb{B}^3$ has sectional curvature $\equiv -1$.

This proves Lemma 6. #

<u>Existence of Thm 1</u>: By Lemmas 5 & 6, we have complete simply-connected Riemannian manifolds of any dimension ≥ 2 with constant sectional curvature $= \pm 1$. By scaling, we have

$$K_{\frac{1}{\sqrt{c}} g} = c K_g \qquad (\forall \text{ metric } g \quad (\text{Ex}))$$

$$= \pm c$$

Together with $\mathbb{R}^n$, we've proved the existence part of Thm 1.

#

§5.2 Geodesic & curvatures

Let $\mathbb{H}^n = \left(\mathbb{B}^n, \frac{4}{(1-|x|^2)^2} \sum_{i=1}^n dx^i \otimes dx^i\right)$.

<u>Facts</u>: $\mathbb{R}^2 \hookrightarrow \mathbb{R}^n$, $S^2 \hookrightarrow S^n$, $\mathbb{H}^2 \hookrightarrow \mathbb{H}^n$ are totally geodesic submanifolds, the studies of geodesics on $\mathbb{R}^n$, S^n & $\mathbb{H}^n$ can be reduced to $\mathbb{R}^2$, S^2, & $\mathbb{H}^2$ (since for any $x, y \in \mathbb{R}^n, S^n$ or $\mathbb{H}^n$, $\exists$ isometry of $\mathbb{R}^n$, S^n or $\mathbb{H}^n$ respectively, taking x to y. (Ex))

Let $M = \mathbb{R}^2, S^2$, or $\mathbb{H}^2$ & $O \in M$ be a fixed point.

Let $C(r) = \{x \in M : d(O, x) = r\}$ be the geodesic circle of radius r.

If $r>0$ small enough, then

$$C(r) = \exp_0 (\text{circle of radius } r \text{ in } T_0M)$$

Denote:

$$\text{length } C(r) = \begin{cases} C_0(r), & \text{if } M=\mathbb{R}^2 \\ C_+(r), & \text{if } M = S^2 \\ C_-(r), & \text{if } M = \mathbb{H}^2 \end{cases}$$

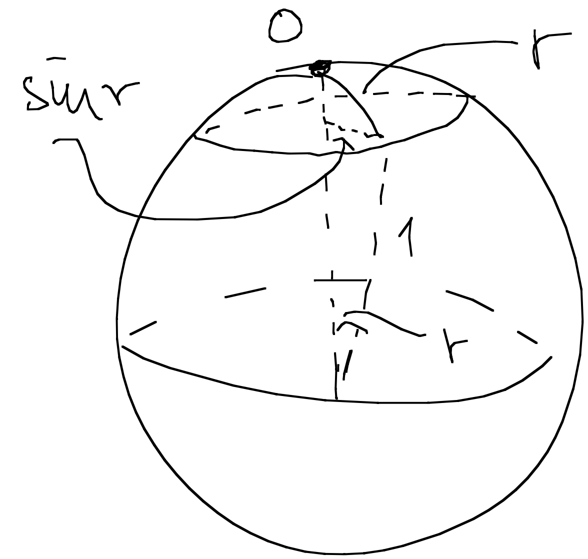
If $M=\mathbb{R}^2$, it is clear that

$$\boxed{C_0(r) = 2\pi r}$$

If $M=S^2$, we may assume 0 = north pole.

Then geodesic circle

$C(r)$ = a circle of radius $\sin r$ in $\mathbb{R}^3$

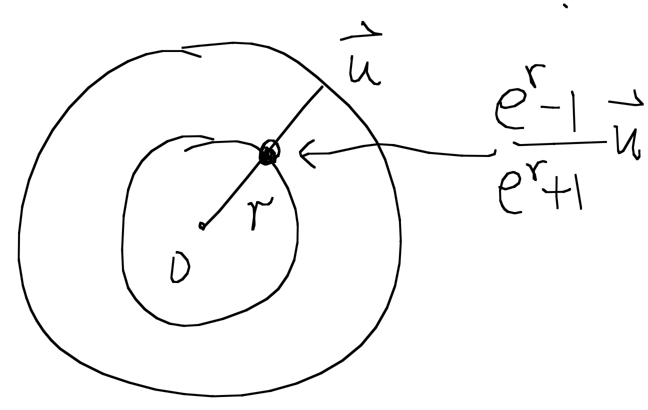


$\Rightarrow$ $\boxed{C_+(r) = 2\pi \sin r}$

If $M = \mathbb{H}^2$, then by the proof of lemma 6, a normal geodesic from O is given by

$$\gamma(s) = \frac{e^s - 1}{e^s + 1}\vec{u}, \quad \text{where } \vec{u} = \text{unit vector in } \mathbb{R}^2.$$

$$\left(|\gamma'(s)|_{\mathbb{H}^2} = 1 \right)$$



$$\Rightarrow \quad d_{\mathbb{H}^2}(0, \gamma(r)) = \int_0^r |\gamma'(s)|_{\mathbb{H}^2}\, ds = r$$

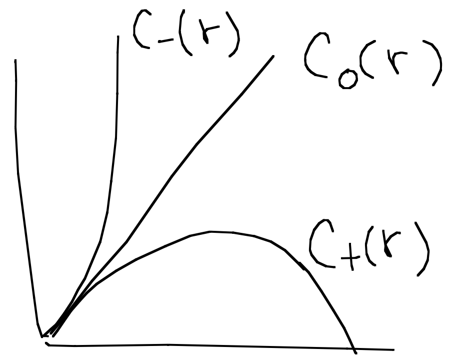
$\Rightarrow$ $C(r)$ = Euclidean circle of radius $\frac{e^r-1}{e^r+1}$ $\left(= \tanh\frac{r}{2}\right)$

$$\Rightarrow \quad C_-(r) = \int_0^{2\pi} \frac{2}{1-\rho^2}\cdot \rho\, d\theta \quad \text{where } \rho = \tanh\frac{r}{2}$$

$$= 2\pi \cdot \frac{2\rho}{1-\rho^2}$$

$$\Rightarrow \quad \boxed{C_-(r) = 2\pi \sinh r}$$

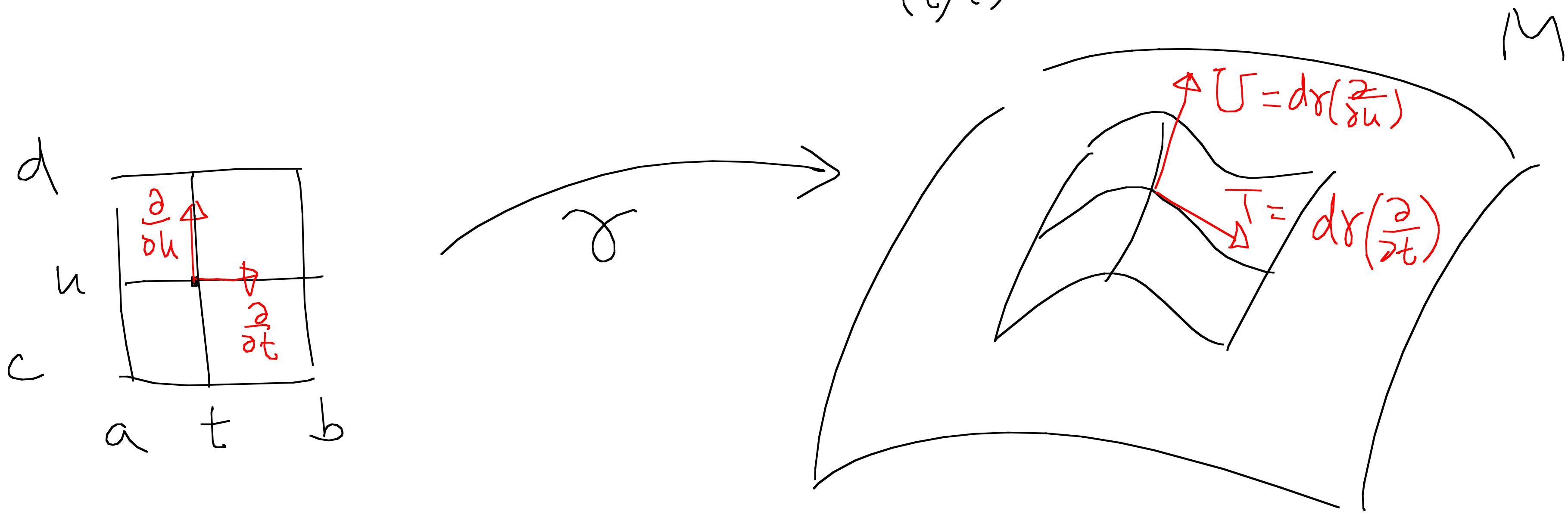
In summary, we have
$$\begin{cases} C_0(r) = 2\pi r \\ C_+(r) = 2\pi \sin r \\ C_-(r) = 2\pi \sinh r \end{cases}$$



To generalize the above to arbitrary complete Riem. manifold, we need to study variations of geodesic.

Let $\gamma : [a,b] \times [c,d] \to M$ be a C^∞ map from the rectangle $[a,b] \times [c,d]$ to a complete Riem manifold M (of any dimension ≥ 2). Denote a point in $[a,b] \times [c,d]$ by (t, u). Then we can define 2 tangent vector fields along γ by

$$\begin{cases} T(t,u) = d\gamma\left(\dfrac{\partial}{\partial t}\Big|_{(t,u)}\right) \\ U(t,u) = d\gamma\left(\dfrac{\partial}{\partial u}\Big|_{(t,u)}\right) \end{cases}$$



$\forall$ fixed $u \in [c,d]$, a curve

$\gamma_u : [a,b] \to M : t \mapsto \gamma(t,u)$ is defined.

Suppose $0 \in [c,d]$. Then γ_0 is called the base curve of γ.

If γ_u are geodesics $\forall u \in [c,d]$, we call γ a one-parameter family of geodesics.

In this case, the vector field $T = \gamma_u'$ and hence

$$D_T T = 0.$$

We also have $[T, U] = d\gamma\left(\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial u}\right]\right) = 0$.

Hence

$$\begin{cases} [T, U] = 0 \\ D_T T = 0 \end{cases} \quad \text{along } \gamma.$$

Then $D_T D_T U = D_T (D_U T)$

$$= D_T D_U T - D_U \cancel{D_T T}^{0} - D_{\cancel{[T,U]}^{0}} T$$

$$= -R_{TU} T$$

Therefore, along the base geodesic γ_0, we have

$$\boxed{D_{\gamma_0'} D_{\gamma_0'} U + R_{\gamma_0' U} \gamma_0' = 0} \quad (\mathrm{Jac})$$

or simply $\boxed{U'' + R_{\gamma_0' U} \gamma_0' = 0}$

where $U'' = D_{\gamma_0'} D_{\gamma_0'} U$ (similarly $U' = D_{\gamma_0'} U$)

Def: • Equation (Jac) is called the Jacobi equation along γ_0.

• Solutions of (Jac) are called Jacobi fields along γ_0.

Note: The vector field U constructed above is called a transversal vector field (or variational vector field) of $\{\gamma_u\}$.

Hence, we have

Lemma 7: A transversal vector field of a 1-parameter family of geodesics is a Jacobi field.

eg: If M = 2 dim'l complete Riem. manifold.

Denote $C(r) = \{ x \in M : d(x, o) = r \}$

$c(r) = \text{length } C(r)$, where $o \in M$ is fixed.

Let (ρ, θ) = polar coordinates on T_oM.

Let $\delta > 0$ small s.t. $\exp_o$ is a diffeomorphism on

$$B(\delta) = \{ v \in T_oM : \rho(v) < \delta \}.$$

We can parametrize a circle of radius r in $B(\delta)$ (centered at o)

by

$$\begin{aligned} \tilde{\gamma} : [0, 2\pi] &\to B(\delta) \\ \theta &\mapsto (r, \theta) \end{aligned}$$

Then $\gamma(r) = \exp_o(\tilde{\gamma})$ and

$$c(r) = \int_0^{2\pi} \left| (d\exp_o)_{(r,\theta)} \left(\frac{\partial}{\partial\theta}\right) \right| d\theta$$

The fact is:

$(d\exp_o)_{(r,\theta)}\left(\frac{\partial}{\partial\theta}\right)$ is a tranversal vecta field (of the family of radial geodesics) with specific initial values.

General setting (for this fact):

Let • M = complete Riem. manifold of dim $n \geq 2$

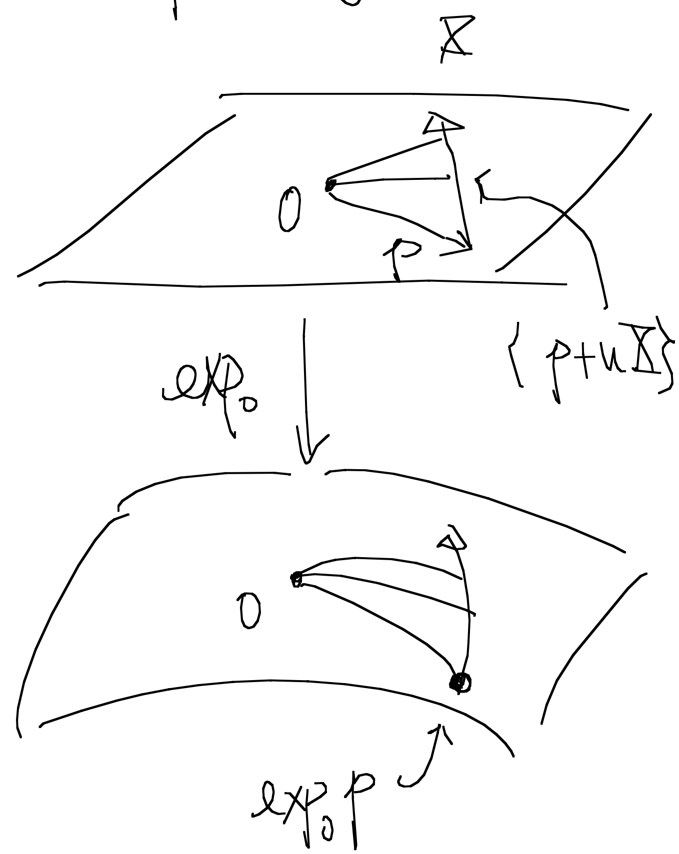
- $0 \in M$ fixed point.

- $p \in T_0M$

- $X \in T_p(T_0M) \cong T_0M$

Define $\Gamma : [0,r] \times [0,1] \to M$, where $r = |p|$ by

$$\Gamma(t,u) = \exp_0\left[\frac{t}{r}(p+uX)\right]$$



Then $\forall u \in [0,1]$, $\Gamma_u(t) = \Gamma(t,u)$ is a geodesic (with initial tangent vector $\frac{1}{r}(p+uX)$).

$\Rightarrow$ $\Gamma(t,u)$ is a 1-para. family of geodesics,

Let $V(t)$ = Transversal vector field along Γ_0,

and $\delta > 0$ be s.t. $\exp_o$ is a diffeo. on

$$B(\delta) = \{ v \in T_oM : |v| < \delta \} \qquad \left(|v| = \rho(v) \right)$$

in polar coordinate

Set $B_\delta = \{ x \in M : d(o,x) < \delta \}$.

Then $B_\delta = \exp_o(B(\delta))$.

let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of T_oM &

$\{\alpha^1, \cdots, \alpha^n\}$ be the dual basis of $\{e_1, \cdots, e_n\}$

Then $\{\alpha^1, \cdots, \alpha^n\}$ are coordinate functions on T_oM.

Define a coordinate system on B_δ by

$$x^i = \alpha^i \circ \exp_{p_0}^{-1} : B_\delta \to \mathbb{R}$$

Then we have

<u>Claim</u>: $\begin{cases} \left\langle \frac{\partial}{\partial x^i}\Big|_0, \frac{\partial}{\partial x^j}\Big|_0 \right\rangle = \delta_{ij}, & \forall i,j \\ D_{\frac{\partial}{\partial x^i}\big|_0} \frac{\partial}{\partial x^j} = 0, & \forall i,j \end{cases}$

(Note: coordinate systems satisfying these conditions are called <u>normal coordinate systems</u>.)

Pf: The 1^{st} eqt. is clearly follows from:

$$(d\exp_0)_0 = \mathrm{Id}.$$

$0 \in M$ $\quad$ $0 \in T_0M$

To see the 2^{nd}, we define a bilinear form

$$\beta : T_0M \times T_0M \longrightarrow \mathbb{R}$$

by $$\beta(e_i, e_j) = D_{\frac{\partial}{\partial x^i}\big|_0} \frac{\partial}{\partial x^j}$$

Then $\forall v = \sum v^i e_i \in T_0M$,

$$\beta(v,v) = \sum_{i,j} v^i v^j D_{\frac{\partial}{\partial x^i}\big|_0} \frac{\partial}{\partial x^j} = D_{\sum v^i \frac{\partial}{\partial x^i}\big|_{i=0}} \left(\sum v^j \frac{\partial}{\partial x^j}\right)$$

Note that $\sum v^j \frac{\partial}{\partial x^j}\Big|_0$ is the initial tangent vector of the geodesic $\exp_0(t \sum v^i e_i)$. Hence $\beta(v,v)=0$ by the geodesic eqt.

$\Rightarrow \beta \equiv 0$ on T_0M

i.e. $D_{\frac{\partial}{\partial x^i}\big|_0} \frac{\partial}{\partial x^j} = 0 \quad \forall i, j$

(This completes the proof of the claim) ※

Now assume $p = \sum p^i e_i$, $X = \sum X^i e_i$ (under $T_p(T_0M) \simeq T_0M$)

For $\varepsilon > 0$ small, $\varepsilon p, \varepsilon X \in B(\delta)$.

Then in the above coordinate system $\{x^1, \cdots, x^n\}$, the coordinate vector of $\Gamma(t,u) = \frac{t}{r}(\vec{p} + u\vec{X})$,

where $\vec{p} = (p^1, \dots, p^n)$ & $\vec{X} = (X^1, \dots, X^n)$,

for $(t,u) \in [0, \varepsilon r] \times [0, \varepsilon]$

And the base geodesic is $\Gamma_0(t) = \Gamma(t,0)$

(in coordinate) $= \frac{t}{r}\vec{p}$

$$\Rightarrow \quad U(t) = \frac{\partial}{\partial u}\Gamma(t,u)$$

(in coordinate) $= \frac{t}{r}\vec{X}$

i.e. $\quad U(t) = \frac{t}{r}\sum X^i \frac{\partial}{\partial x^i}\Big|_{(t,0)}$

Therefore $U(0) = 0$, and

$$U'(0) = D_{\Gamma_0'(0)} U = \frac{d}{dt}\Big|_{t=0} \left(\frac{t}{r} \sum X^i \frac{\partial}{\partial x^i}\Big|_{(t,0)} \right)$$

$$= \frac{1}{r} \sum X^i \frac{\partial}{\partial x^i}\Big|_0 + 0$$

In conclusion, the tranversal vector field $U(t)$ of $\Gamma(t,u) = \exp_{p_0}\left[\frac{t}{r}(p + uX)\right]$ satisfies

$$\begin{cases} U(0) = 0 \\ U'(0) = \frac{1}{r}\vec{X} \text{ (in coordinate)}, \end{cases}$$

where $r = |p|$.

$$\left[\text{Recall: } \; V(t) = \frac{t}{r}\left(d\exp_o\right)_{\exp_o\left(\frac{t}{r}p\right)}(X) \qquad (\text{check!}) \right]$$

Applying the above to $M = \mathbb{R}^2, S^2$ a $\mathbb{H}^2$ with

$p = (r, \theta)$, $X = \frac{\partial}{\partial \theta}\Big|_{(r,\theta)}$.

Therefore $V(r) = \left(d\exp_o\right)_{(r,\theta)}\left(\frac{\partial}{\partial\theta}\right)$ (at $t = r$)

is a Jacobi field satisfying

$$\begin{cases} V(0) = 0 \\ |V'(0)| = \frac{1}{r}\left|\frac{\partial}{\partial\theta}\right| = 1. \end{cases} \quad \left((r,\theta) = \text{polar coordinates}\right)$$

Let $W(t)$ = unit parallel vector field along Γ_0 s.t.

$$\langle W(t), \Gamma_0'(t) \rangle = 0.$$

On the other hand,

Gauss lemma $\Rightarrow$ $U(t) = (d\exp_o)_{(t,0)}\left(\frac{\partial}{\partial\theta}\right)$

is normal to $\Gamma_0'(t)$.

In our case of $\dim M = 2$,

$$U(t) = (d\exp_o)_{(t,0)}\left(\frac{\partial}{\partial\theta}\right) = f(t)\,W(t).$$

for some function $f \in C^\infty[0,r]$.

Then $D_{\Gamma_0'(t)} U(t) = f'(t)\,W(t)$

$\Rightarrow$ $$D_{\Gamma_0'(t)} D_{\Gamma_0'(t)} U(t) = f''(t)\, W(t)$$
(since W is parallel)

Now (Jac) $\Rightarrow$

$$f''(t) W(t) + R_{\Gamma_0', fW} \Gamma_0' = 0$$

$$\Leftrightarrow \quad f''(t) + \langle R_{\Gamma_0' W} \Gamma_0', W \rangle f = 0$$

$$\Rightarrow \quad f'' + K f = 0$$

where K = Gauss curvature at $\Gamma_0(t)$

(since $|\Gamma_0'(t)| = |W(t)| = 1$ & $\langle \Gamma_0', W \rangle = 0$)

We may also assume $\langle W, \frac{\partial}{\partial \theta} \rangle > 0$, we have

$$\begin{cases} f'' + Kf = 0 \\ f(0) = 0 \\ f'(0) = 1 \end{cases}$$

$\therefore$ The signature of K has implication on

$$C(r) = \int_0^{2\pi} \left| (d\exp_o)_{(r,\theta)} \left(\frac{\partial}{\partial \theta} \right) \right| d\theta$$

In particular, if $K \equiv 0, +1, -1$ we have

$$f(r) = \begin{cases} r & , K \equiv 0 \\ \sin r & , K \equiv +1 \\ \sinh r & , K \equiv -1 \end{cases}$$