Ch4 Exponent Map, Gauss Lemma, & Completeness

Let
- M = Riemannian manifold with metric
- $g = g_{ij}\, dx^i \otimes dx^j$ $\quad (g = \langle\ ,\ \rangle)$
- D = Levi-Civita connection of g

4.1 Exponent map

Recall: $\gamma : [0, L] \to M$ is a geodesic (wrt D)

$$\iff \quad D_{\gamma'} \gamma' = 0$$

Facts:
- If γ is a geodesic, $|\gamma'|$ is a constant.
- If $\gamma : [0, L] \to M$ is a geodesic,

then $\forall$ constant $c > 0$,

$$\gamma^c : [0, \tfrac{L}{c}] \longrightarrow M : t \mapsto \gamma(ct)$$

is also a geodesic, and

$$|(\gamma^c)'| = c|\gamma'|$$

Therefore, we can <u>normalize</u> our geodesic to have $|\gamma'| = 1$.

<u>Recall</u>: If $\xi : [a,b] \to M$ is a C^∞ curve, then the <u>length</u> of ξ is defined by

$$L(\xi) = \int_a^b |\xi'| dt.$$

If ξ is <u>regular</u>, i.e. $|\xi'(t)| > 0$, $\forall t \in [a,b]$,

then $s(t) = \int_a^t |\xi'(\tau)| d\tau = L(\xi|_{[a,t]})$

defines a C^∞ function $s : [a,b] \to [0, L(\xi)]$

with $\frac{ds}{dt} = |\xi'(t)| > 0$

Hence $u = s^{-1} : [0, L(\xi)] \to [a,b]$ exists & C^∞

And $\tilde{\xi}(s) \underline{\underline{\mathrm{def}}}\ \xi(u(s)) : [0, L(\xi)] \to M$

is a <u>reparametrization</u> of ξ such that

$$\left|\frac{d\tilde{\xi}}{ds}\right| = 1$$

Terminology:

- s = arc-length parameter
- ξ is said to be parametrized by arc-length
- a normalized geodesic is a geodesic parametrized by arc-length

  i.e. $D_{\gamma'}\gamma' = 0$ & $|\gamma'| = 1$

Note: All the above can be extended to piecewise C^1 curve.

Recall: $D_{\gamma'}\gamma' = 0$ is a (nonlinear) ODE system and hence we have the following result by applying the theory of ODE:

<u>Thm</u>: $\forall\, x \in M$ & $\varepsilon > 0$

$\exists$ nbd $\mathcal{U}$ of x, and $\delta > 0$

such that

$\forall\, y \in \mathcal{U}$ and $v \in T_y M$ with $|v| < \delta$,

$\exists$ unique geodesic $\gamma_v : I \to M$,

defined on an open interval I containing

$[-\varepsilon, \varepsilon]$, with initial condition

$$\begin{cases} \gamma_v(0) = y \\ \gamma_v'(0) = v \end{cases}.$$

If γ_v is a geodesic by above, then

$$\xi_v(t) \overset{\text{def}}{=} \gamma_v(\varepsilon t)$$

is a geodesic defined on an open interval containing $[0,1]$. Therefore, we have

<u>Thm(#)</u> $\forall x \in M$, $\exists$ nhd. $\mathcal{U}$ of x and $\omega > 0$ s.t.

$\forall y \in \mathcal{U}$ and $v \in T_yM$ with $|v| < \omega$, $\exists$ unique geodesic $\gamma_v : I \to M$ defined on an open interval I containing $[0,1]$ with initial conditions

$$\gamma_v(0) = y \quad \& \quad \gamma_v'(0) = v.$$

<u>Def</u>: Let $\omega > 0$ be given in Thm (#). The <u>exponential map</u> $\exp_x$ <u>at x</u>, defined on

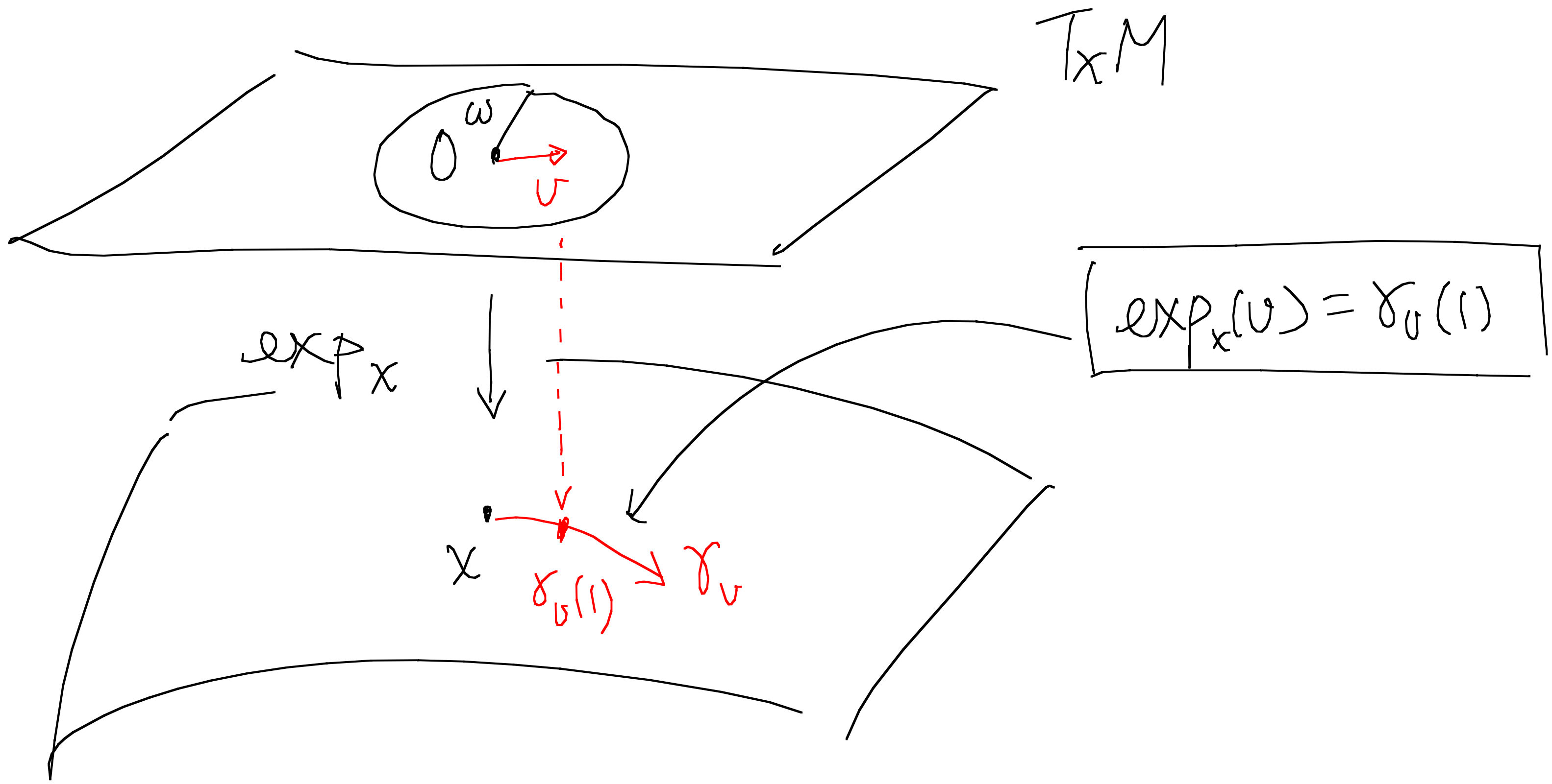
$$B_x(\omega) = \{v \in T_xM : |v| < \omega\} \subset T_xM,$$

is the map

$$\exp_x : B_x(\omega) \to M : v \longmapsto \gamma_v(1)$$

where γ_v is given by Thm (#).

That is, $\boxed{\exp_x(v) = \gamma_v(1)}$.



<u>Fact</u>: Let $\mathcal{V} = \{ (y,v) \in TM : y \in \mathcal{U},\ |v| < \omega \} \subset TM$

(with $\mathcal{U}, \omega$ as in Thm(#)) Then Thm(#)

$$\Rightarrow \quad \exp(y,v) \overset{\text{def}}{=\!=} \exp_y(v)$$

defines a map $\exp : \mathcal{V} \longrightarrow M$.

By ODE theory (& Thm(#)), $\exp : U \to M$ is C^∞.

In particular $\exp_x : B_x(\omega) \to M$ is C^∞.

(Pf : See Gallot, Hulin, & Lafontaine)

Note: In fact, we can show that

$$\exp_x : \mathcal{B} \to M \in C^\infty$$

on the maximal domain of the definition of $\exp_x$.

Note: In the case of

$M = SO(n,\mathbb{R}) = \{ A \text{ } n\times n \text{ matrix} : A^T A = I, \det A = 1 \}$

with metric defined by $(n-2)\operatorname{tr}(XY)$ for

$X, Y \in so(n, \mathbb{R}) = T_{Id} M = \{ B \text{ } n\times n \text{ matrix} : B^T + B = 0 \}$.

Then $\exp_{Id} : T_{Id} M \to M$ is given by

$$\exp_{Id} B = e^B = \sum_{k=0}^{\infty} \frac{B^k}{k!},$$

$\forall B \in T_{Id} M = \{ B^T + B = 0 \}$.

This is the reason for the terminology.

<u>Thm</u>: $\exp_x$ is a diffeomorphism in a nbd of $0 \in T_x M$.

This Thm follows immediately from

<u>Lemma</u>: $(d \exp_x)_0 =$ "identity of $T_x M$".

Note: $\exp_x : B(\omega) \subset T_xM \longrightarrow M$ with $\exp_x(0) = x$.

Therefore $(d\exp_x)_0 : T_0(T_xM) \to T_xM$

Since T_xM is linear,

$$T_0(T_xM) \cong T_xM$$

(In fact, $\forall\, v \in T_xM$, we define

$\xi_v : t \mapsto tv$ a curve in T_xM

with $\xi_v(0) = 0$ & "$\xi_v'(0) = v$")

Hence $(d\exp_x)_0$ can be regarded as a map from T_xM to itself.

Pf of lemma : $\forall\ v \in T_xM \cong T_0(T_xM)$

$$(d\exp_x)_0(v) = \frac{d}{dt}\Big|_{t=0} \exp_x(tv)$$ (identification of $T_0(T_xM) \cong T_xM$)

$$= \frac{d}{dt}\Big|_{t=0} \gamma_{tv}(1)$$ (definition of $\exp_x$)

$$= \frac{d}{dt}\Big|_{t=0} \gamma_v(t)$$ (ex.)

$$= \gamma_v'(0) = v$$ ※

We can even have a stronger result :

<u>Thm</u>: $\forall$ compact $K \subset M$, $\exists\, \varepsilon > 0$ s.t.

$\forall x \in K$, $\exp_x$ is diffeo on $B_x(\varepsilon)$.

(This shows that we can find a <u>uniform</u> ε $\forall$ cpt. $K \subset M$)

<u>Pf</u>: It is sufficient to show that

$$\boxed{\begin{array}{l}\forall x \in M, \exists\, \varepsilon > 0, \text{ \& open nhd. } \Omega \text{ of } x \text{ s.t.} \\ \forall y \in \Omega, \exp_y \text{ is a diffeo. on } B_y(\varepsilon) \subset T_yM,\end{array}}$$

By Thm(#), $\exists$ nhd $\mathcal{U}$ of x s.t.

$\exp_y$ is defined on some ball $B_y(\varepsilon(y))$, $\varepsilon(y) > 0$.

Let $N = \{(y, v) : y \in \mathcal{U}, v \in B_y(\varepsilon(y))\} \subset TM$, and define

$$\begin{aligned} E: N &\longrightarrow M \times M \\ \rotatebox{90}{$\in$} \quad & \qquad\quad \rotatebox{90}{$\in$} \\ (y, v) &\longmapsto (y, \exp_y v) \end{aligned}$$

By theory ODE, E is C^∞.

Choose a coordinate system $\{x^1, \dots, x^n\}$ centered at x (i.e. $x^i(x) = 0$). Then any (y, v) can be represented by coordinates $(x^1, \dots, x^n, u^1, \dots, u^n)$

where $\{u^i\}$ are given by $v = \sum u^i \frac{\partial}{\partial x^i}$.

(i.e. $u^i = dx^i(v)$, $\forall i$)

$\Rightarrow \left\{ \frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial u^1}, \cdots, \frac{\partial}{\partial u^n} \right\}$ is a basis of the tangent space $T_{(y,u)}(TM)$ of TM.

Now

$$dE_{(x,0)}\left(\left.\frac{\partial}{\partial x^i}\right|_{(x,0)} \right) = \left.\frac{d}{dt}\right|_{t=0} E(\xi_i(t), 0)$$

where $\xi_i(t)$ is a curve in M s.t.

$$\xi_i(0) = x \quad \& \quad \xi_i'(0) = \frac{\partial}{\partial x^i}$$

(i.e. $t \mapsto (\xi_i(t), 0)$ curve in TM)

$$\Rightarrow \quad dE_{(x,0)}\left(\left.\frac{\partial}{\partial x^i}\right|_{(x,0)} \right) = \left.\frac{d}{dt}\right|_{t=0} \left(\xi_i(t), \exp_{\xi_i(t)}(0) \right)$$

$$= \frac{d}{dt}\Big|_{t=0} \left(\xi_i(t), \xi_i(t) \right)$$

$$= \left(\frac{\partial}{\partial x^i}\Big|_x , \frac{\partial}{\partial x^i}\Big|_x \right)$$

Also $dE_{(x,0)} \left(\frac{\partial}{\partial u^i}\Big|_{(x,0)} \right) = \frac{d}{dt}\Big|_{t=0} E\left(x, t \frac{\partial}{\partial x^i}\Big|_x \right)$

$$= \frac{d}{dt}\Big|_{t=0} \left(x, \exp_x\left(t \frac{\partial}{\partial x^i} \right) \right)$$

$$= \left(0, (d\exp_x)_0 \left(\frac{\partial}{\partial x^i} \right) \right)$$

$$= \left(0, \frac{\partial}{\partial x^i}\Big|_x \right) \quad \text{by previous lemma.}$$

$\Rightarrow \quad dE_{(x,0)} : T_{(x,0)}N \to T_xM \times T_xM$ is nonsingular.

$\therefore$ IFT $\Rightarrow$ E is a local diffeo. that maps a nbd $\mathcal{W}$ of $(x,0)$ in TM to a nbd of

$$(x, \exp_x(0)) = (x,x) \text{ in } M \times M.$$

Therefore, $\exists\, c>0,\ \varepsilon'>0$ s.t.

$$\{(y,v) \in TM : |x^i(y)| \leq c,\ |u^i(v)| \leq \varepsilon'\}$$

is a cpt. subset of $\mathcal{W}$.

$\Rightarrow \exists\, \varepsilon > 0$ s.t.

norm wrt metric g.

$$\{(y,v) \in TM : |x^i(y)| \leq c,\ |v| \leq \varepsilon\} \subset \mathcal{W}$$

Then this $\varepsilon>0$, & $\Omega=\{y\in\mathcal{U}: |x^i(y)|\le c\}$ satisfy the requirement. #

4.2 Gauss Lemma, minimizing geodesic.

Let (M,g) be a Riemannian manifold and $x\in M$ be fixed. Let $\delta>0$ sufficiently small such that $\exp_x$ is a diffeomorphism on $B(\delta)=\{v\in T_xM: |v|<\delta\}$, where $|v|=\langle v,v\rangle^{1/2}$. Denote

$$B_\delta=\exp_x(B(\delta))$$

Then

- $\gamma(t) = \exp_x(tv)$, $t \in [0,1]$, $v \in B(\delta)$

  is called a radial geodesic (segment)

  joining x to $\exp_x(v)$.

And $\forall\, t \in (0, \delta)$,

- $S_t = \exp_x(\{v \in T_xM : |v| = t\})$ is called the geodesic sphere of radius t centered at x.

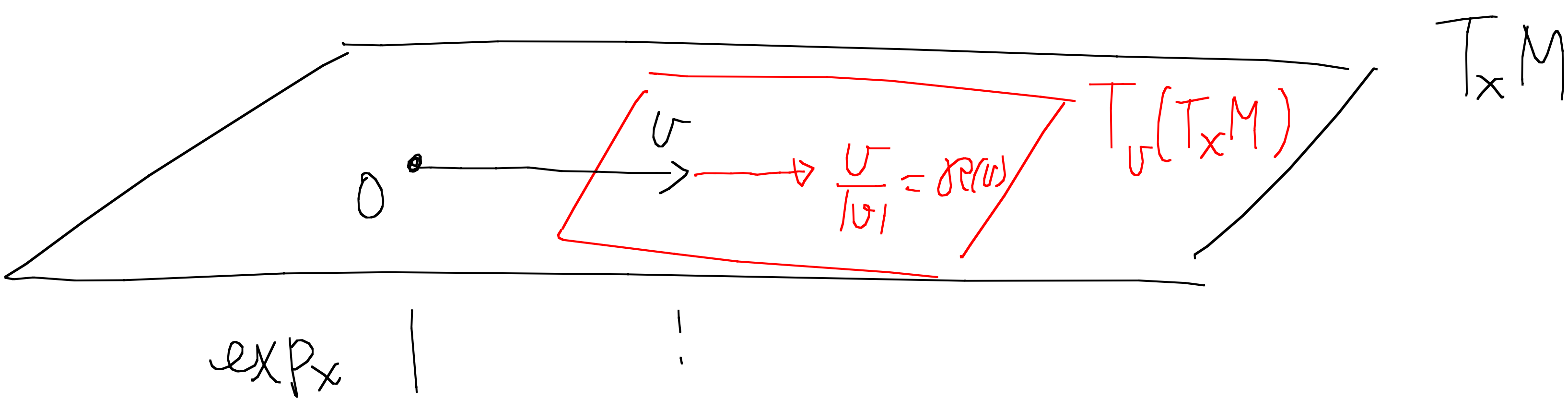
- $B_t = \exp_x(B(t))$ is called the geodesic ball of radius t centered at x.

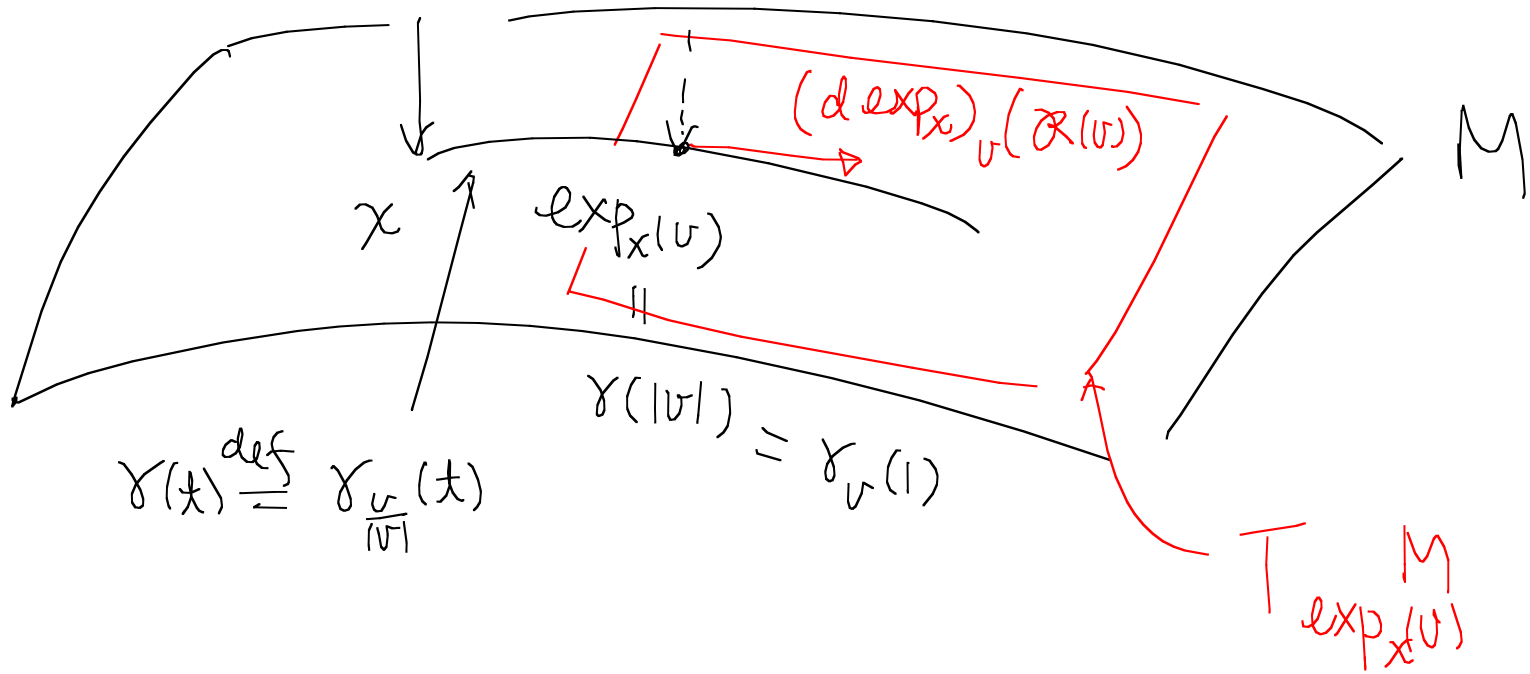
<u>Lemma</u>: (M,g), x, δ as above. Define a vector field $\mathcal{X}$ on $T_xM \setminus \{0\}$ by

$$\mathcal{X}(v) = \frac{v}{|v|} \qquad \left(\mathcal{X} : T_xM\setminus\{0\} \to T\left(T_xM\setminus\{0\}\right) \right)$$

with $T_v(T_xM\setminus\{0\}) \cong T_xM$

Then $$\left| (d\exp_x)_v (\mathcal{X}(v)) \right| = 1.$$





Pf: For $v \in T_xM \setminus \{0\}$, let $\gamma(t) = \gamma_{\frac{v}{|v|}}(t)$ the normalized geodesic on M s.t. $\gamma(0) = x$, $\gamma'(0) = \frac{v}{|v|}$

By definition of $\exp_x$,

$$\exp_x(v) = \gamma(|v|)$$

Since $\mathcal{R}(v) =$ unit tangent vector of the line

$$v + t\,\mathcal{R}(v).$$

$$(d\exp_x)_v(\mathcal{R}(v)) = \frac{d}{dt}\Big|_{t=0} (\exp_x)(v + t\,\mathcal{R}(v))$$

$$= \frac{d}{dt}\Big|_{t=0} (\exp_x)\left((|v| + t)\frac{v}{|v|}\right)$$

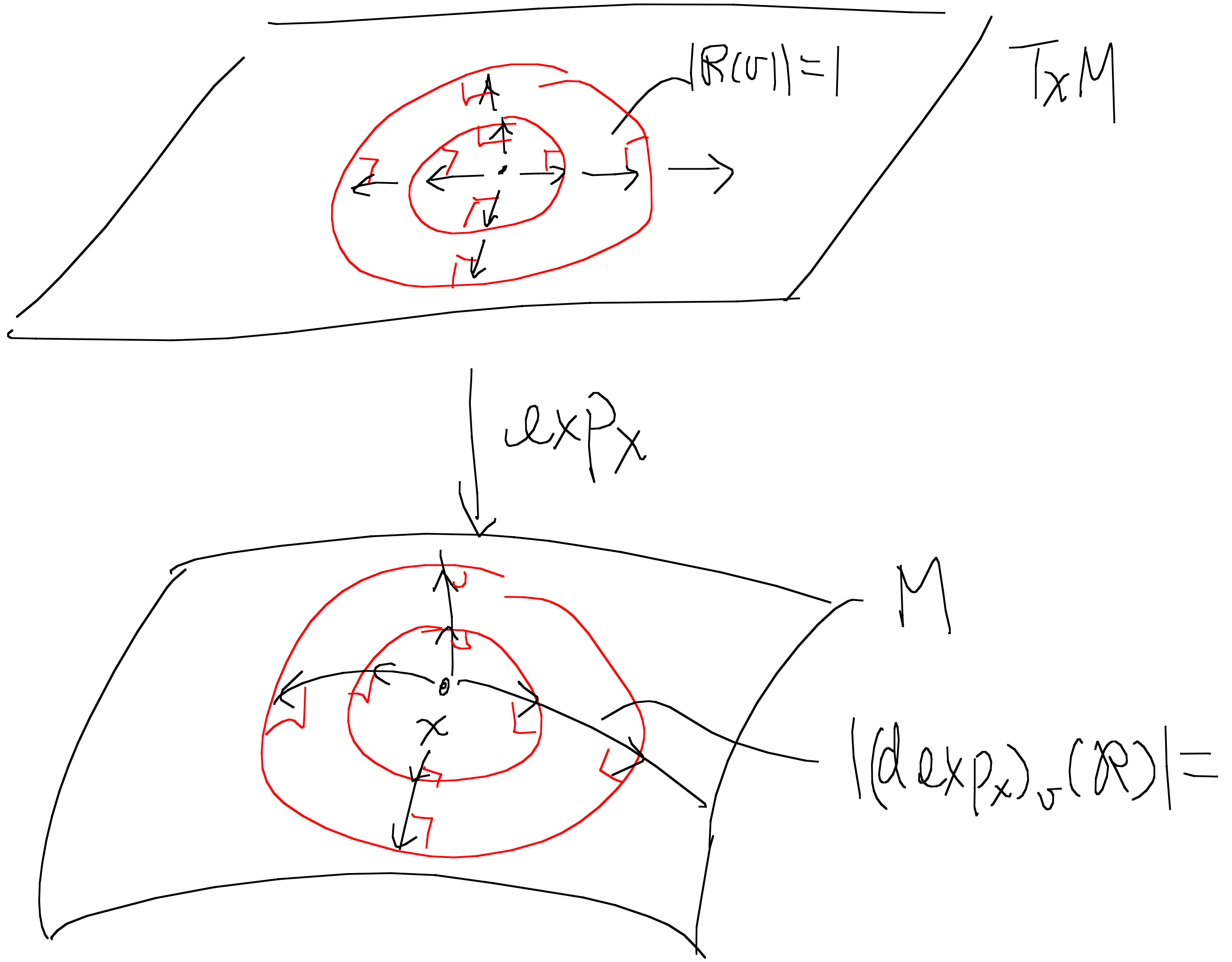
$$= \frac{d}{dt}\Big|_{t=0} \gamma(|v| + t)$$

$$= \gamma'(|v|)$$

$$\therefore \quad \left|(d\exp_x)_v(\mathcal{R}(v))\right| = |\gamma'(|v|)| = |\gamma'(0)| = 1 \quad \#$$

<u>Gauss Lemma</u>: Radial geodesic are <u>athogonal</u> to

the geodesic sphere S_t, $\forall t \in (0,\delta)$.



Pf: Define a diffeo

$$\begin{array}{ccc} F = S^{n-1} \times (0, \delta) & \xrightarrow{\subset T_xM} & B_\delta \setminus \{x\} \\ \cup & & \\ (p, t) & \longmapsto & F(p,t) = \exp_x(tp) \end{array}$$

Then for fixed $t \in (0, \delta)$

$$F(\cdot, t) : S^{n-1} \times \{t\} \longrightarrow S_t$$

is a diffeomorphism.

Let γ = radial geodesic intersecting S_t at the point $\exp_x(tp)$.

We take a local coordinate $\{y^1, \cdots, y^{n-1}\}$ around $p \in S^{n-1}$. And let r be the natural parameter of the interval $(0, \delta)$.

Then $\begin{cases} R = dF\left(\frac{\partial}{\partial r}\right) \\ Y_i = dF\left(\frac{\partial}{\partial y_i}\right) \end{cases}$

are vector fields on $B_\delta \setminus \{x\} \subset M$ s.t.

Y_i are tangential to S_t (and form a basis of $T_y S_t$ (for $y \in S_t \subset B_\delta \setminus \{x\}$))

and R is tangential to a radial geodesic.

Therefore, we need to show that $\langle R, Y_i \rangle = 0 \ \forall i$ at $\exp_x(tp)$.

Consider $\langle R, Y_i \rangle$ along the radial geodesic γ.

Then $\langle R, Y_i \rangle'$ ← derivative wrt parameter of γ (i.e. $r \in (0, \delta)$)

$= R \langle R, Y_i \rangle$

$= \langle D_R R, Y_i \rangle + \langle R, D_R Y_i \rangle$

$= 0 + \langle R, D_{Y_i} R \rangle + \langle R, [R, Y_i] \rangle$

(Since $D_R R = D_{\gamma'} \gamma' = 0$)

However $[R, Y_i] = [dF(\frac{\partial}{\partial r}), dF(\frac{\partial}{\partial y_i})]$ (ex.)

$= dF([\frac{\partial}{\partial r}, \frac{\partial}{\partial y_i}])$

$= 0$

Hence $\langle R, Y_i \rangle' = \langle R, D_{Y_i} R \rangle = \frac{1}{2} Y_i \langle R, R \rangle$

$= 0$ (by lemma that $|R| = 1$)

$\Rightarrow \langle R, Y_i \rangle = \lim_{r \to 0} \langle R, Y_i \rangle (\gamma(r))$

$= 0$ since $|Y_i| \to 0$ as $\gamma(r) \to x$

($S_t \to \{x\}$ as $t \to 0$)

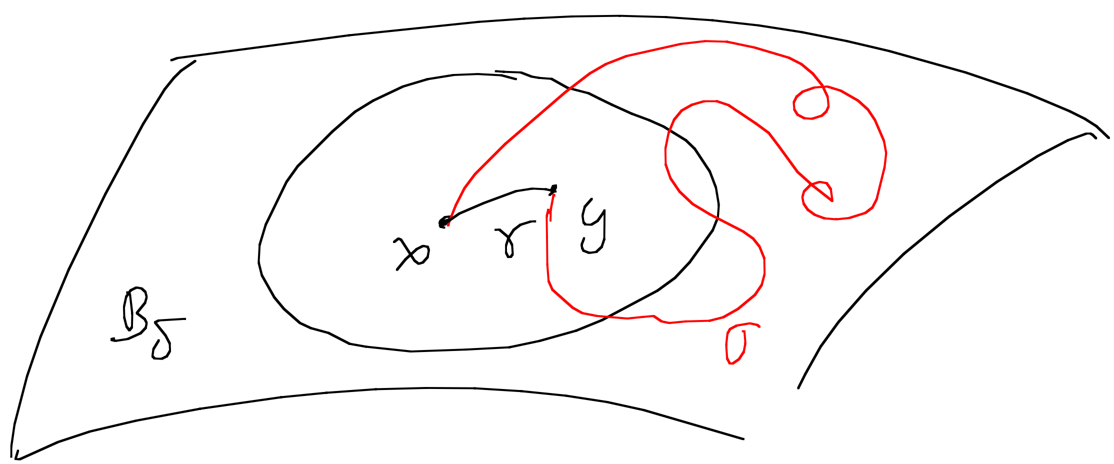
#

Thm: Let
- (M, g) = Riemannian manifold
- $x \in M$
- $\delta > 0$ s.t. $\exp_x : B(\delta) \to B_\delta$ is a diffeo.

- γ = unique radial geodesic joining x and a point $y \in B_\delta \setminus \{x\}$

Then $L(\gamma) \le L(\sigma)$ for all piecewise smooth curve σ on M (not necessary within B_δ) joining x to y.

Equality holds $\iff$ σ = monotonic reparametrization of γ.



Cor: Let $\gamma : [0, c] \to M$ be a arc-length parametrized piecewise smooth curve such that $L(\gamma) \leq L(\sigma)$ $\forall$ piecewise smooth curve σ joining $\gamma(0)$ & $\gamma(c)$. Then γ is a geodesic.

Caution: The converse of the Cor. is not true in general.

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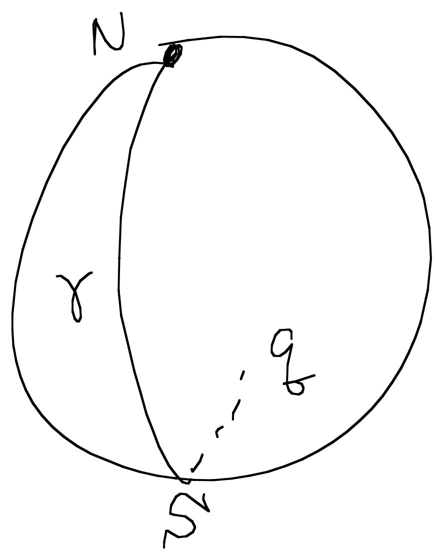

γ = geodesic, but not length minimizing.

Def: A geodesic $\gamma : [0, c] \to M$ is called a <u>minimizing geodesic</u> if $L(\gamma) \leq L(\sigma)$ $\forall \sigma$ joining $\gamma(0)$ & $\gamma(c)$.

<u>Pf of Cor</u> (<u>Assuming the Thm</u>)

Let $x = \gamma(0)$. Choose B_δ as in thm.

Let $t_1 = \min\{t : \gamma(t) \in \partial B_\delta\}$. (If t_1 doesn't exist, then we are done.)

If $\gamma|_{[0,t_1]}$ is not geodesic, then by the thm,

$$L(\gamma|_{[0,t_1]}) > L(\gamma_1)$$

where γ_1 = radial geodesic joining $x = \gamma(0)$ & $\gamma(t_1)$ in B_δ.

$\Rightarrow L(\gamma_1 \cup \gamma|_{[t_1, c]}) < L(\gamma)$

which is a contradiction.

Hence $\gamma|_{[0,t_1]}$ is a geodesic.

Continuing this argument $\Rightarrow \gamma|_{[0,c]}$ is a geodesic.

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Pf: (of Gauss Lemma)

As in the proof of the Gauss Lemma, we can find basis $\{R, Y_1, \cdots, Y_{n-1}\}$ of T_zM for $z \in B_\delta \setminus \{x\}$ s.t. $R =$ tangential to the radial direction & $Y_1, \cdots, Y_{n-1} =$ tangential to the geodesic sphere.

WLOG, we may assume $\sigma \subset B_\delta$.

Then for any such $\sigma : [0,1] \to B_\delta$ s.t.

$$\sigma(0) = x \quad , \quad \sigma(1) = y,$$

we have $\forall t \in [0,1]$

$$\sigma'(t) = f(t) R(\sigma(t)) + T(t)$$

for some function $f(t)$, where

$$T(t) = \text{a linear combination of } Y_i\text{'s}.$$

Let $v \in B(\delta)$ be the unique vector s.t.

$$\exp_x(v) = y$$

Then $\quad \xi = \exp_x^{-1} \circ \sigma \quad$ is a curve in $B(\delta) \subset T_x M$

joining 0 and v.

Since $(d\exp_x^{-1})(R) = \mathcal{R}$ $\Big($ = unit radial vector field defined above.

$(d\exp_x^{-1})(Y_i)$ tangential to $S^{n-1}_{|\xi(t)|} \subset T_x M$, we see that

$$(d\exp_x^{-1})(\langle \sigma', R\rangle R) = f R$$

is the radial projection of the tangent vector ξ'.

$$\Rightarrow \quad |v| = |\xi(1)| - |\xi(0)| = \int_0^1 f(t)\,dt$$

$$\Rightarrow \quad L(\gamma) = \int_0^1 f(t)\,dt \quad \left(\text{since } \gamma \text{ is the radial geodesic}\right)$$

Gauss Lemma $\Rightarrow$

$$|\sigma'(t)|^2 = f^2(t)\,|R(\sigma(t))|^2 + |T(t)|^2$$

$$= f^2(t) + |T(t)|^2$$

$$\Rightarrow \quad L(\sigma) = \int_0^1 |\sigma'|$$

$$= \int_0^1 \sqrt{f^2(t) + |T(t)|^2}\, dt$$

$$\geq \int_0^1 f(t)\, dt = L(\gamma).$$

Finally, if $L(\sigma) = L(\gamma)$, then $T(t) = 0$ & $f > 0$.

$$\Rightarrow \quad \sigma'(t) = f(t) R(\sigma(t)) \quad \text{with } f > 0$$

$\Rightarrow$ σ = monotonic reparametrization of γ. $\#$