2.3 Geodesic

Def: A curve $\gamma : [a,b] \to M$ is called a geodesic wrt the connection D if $\gamma'(t)$ is parallel along γ.

In local coordinates $\{x^i\}$

$$\gamma(t) = (x^1(t), \dots, x^n(t))$$

$$\Rightarrow \quad \gamma'(t) = \sum \frac{dx^i}{dt}(t) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)}$$

Hence

$$D_{\gamma'(t)} \gamma'(t) = \sum_k \left[\frac{d}{dt}\left(\frac{dx^k}{dt}\right) + \Gamma^k_{ij}(\gamma(t)) \frac{dx^i}{dt}\frac{dx^j}{dt} \right] \frac{\partial}{\partial x^k}$$

$\therefore$ γ is a geodesic (wrt D) $\Leftrightarrow D_{\gamma'}\gamma' = 0$

$$\Leftrightarrow \quad \frac{d^2 x^k}{dt^2} + \Gamma^k_{ij}(x^1, \dots, x^n)\frac{dx^i}{dt}\frac{dx^j}{dt} = 0 \qquad \forall\, k = 1, \dots, n$$

which is a <u>non-linear</u> ODE system for $(x^1(t), \dots, x^n(t))$.

ODE theory $\Rightarrow$

<u>Lemma</u>: $\forall$ connection D on M;

$\forall\, v \in T_x M$

$\Rightarrow \exists!$ geodesic $\gamma(t)$ wrt D on <u>some interval</u> $(-\varepsilon, \varepsilon)$

s.t. $\gamma(0) = x$ and $\gamma'(0) = v$.

Note: If D is Levi-Civita connection of g.

Then $\forall$ geodesic γ of D, we have

$$\frac{d}{dt}\langle \gamma', \gamma' \rangle = \langle D_{\gamma'}\gamma', \gamma' \rangle + \langle \gamma', D_{\gamma'}\gamma' \rangle = 0$$

$\Rightarrow$ $|\gamma'(t)|$ is a constant.

2.4 Induced connection

Let M = Riemannian manifold

N = differentiable manifold

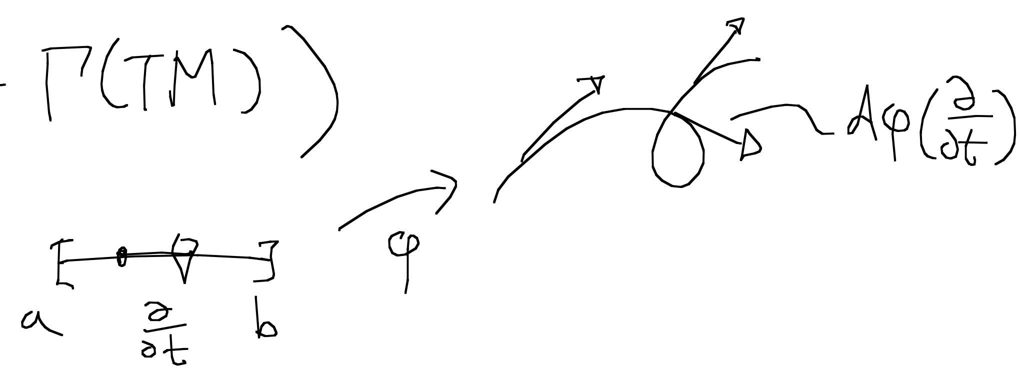
and $\varphi : N \to M$ C^∞ map

Def: A map $X : N \to TM$ is called a <u>vector field along</u> φ if $\forall x \in N$, $X(x) \in T_{\varphi(x)}M$.

$$\begin{tikzcd} & TM \arrow[d] \\ N \arrow[ur, "X"] \arrow[r, "\varphi"'] \arrow[ur, phantom, shift right=3, "\circlearrowright"] & M \end{tikzcd}$$

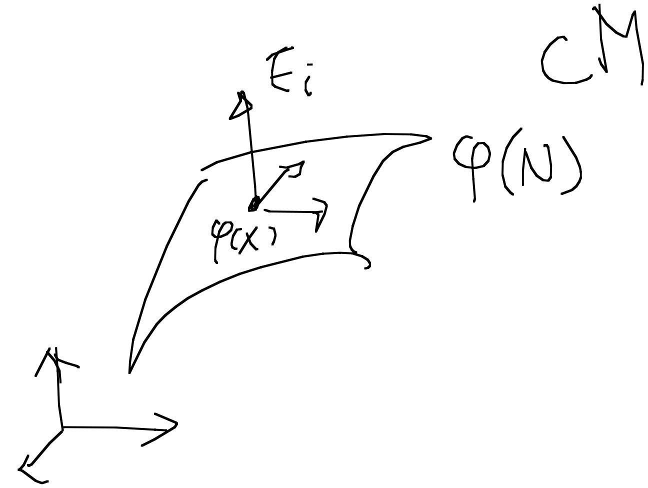
eg: $X \in \Gamma(TN)$, $d\varphi(X)$ is a vector field along φ

(but not necessary $\in \Gamma(TM)$)



<u>Note</u>: If $v \in T_xN$, and $\{E_i\}_{i=1}^{n}$ is "frame field" in a nbd V of $\varphi(x) \in M$

(ie $\{E_i(p)\}$ is a basis of T_pM $\forall\, p$ in V, ($E_i(p)$ smooth in p))



Then $\forall\, x \in \varphi^{-1}(V) \subset N$

$$X(x) = \sum X^i(x)\, E_i(\varphi(x)) \in TM,$$ for some functions $X^i(x)$ on N.

Define

$$\tilde{D}_v X = \sum \left[v(X^i)(x)\, E_i(\varphi(x)) + X^i(x)\, D_{d\varphi(v)} E_i \right]$$

where D = Levi-Civita connection M

Fact: $\tilde{D}_v \mathbb{X}$ is well-defined (indep of the choice of $\{E_i\}$)

<u>Def</u>: • $\tilde{D}$ is called the <u>induced connection</u>

• $\forall$ $V \in \Gamma(TN)$, $\mathbb{X}$ = vector field along φ

$$(\tilde{D}_V \mathbb{X})(x) \overset{\text{def}}{=\!=} \tilde{D}_{V(x)} \mathbb{X}$$

<u>Facts</u>: If D = Levi-Civita on M, then

• $\forall$ $\mathbb{X}, Y \in \Gamma(TN)$

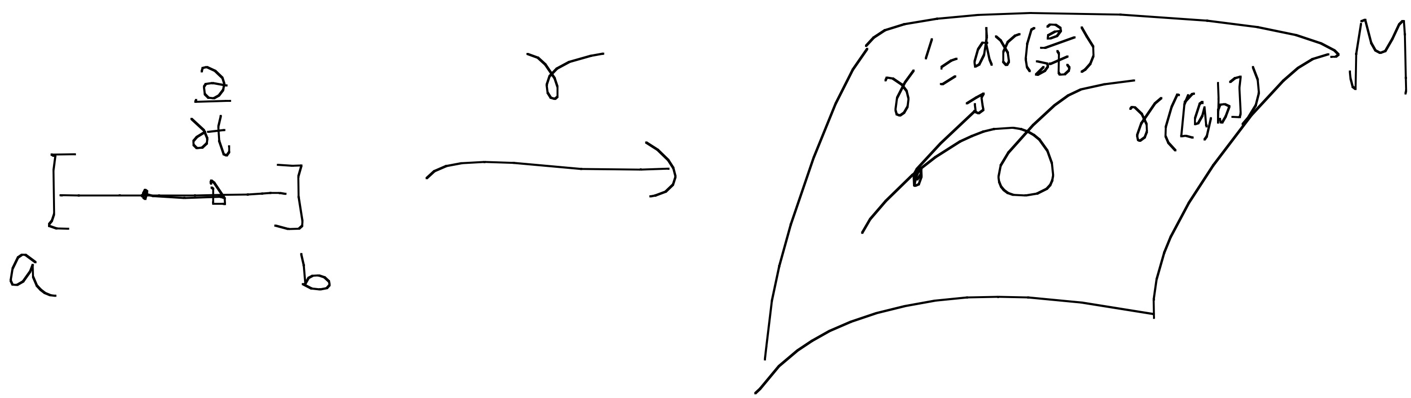
$$\tilde{D}_{\mathbb{X}} d\varphi(Y) - \tilde{D}_Y d\varphi(\mathbb{X}) - d\varphi([\mathbb{X}, Y]) = 0.$$

- $\forall$ V, W vecta fields along φ & $u \in T_x N$,

then $$u\langle V, W\rangle = \langle \tilde{D}_u V, W\rangle + \langle V, \hat{D}_u W\rangle$$

Note: If $\gamma : [0,1] \to M$ is a smooth curve (not necessary embedded) then

$$\gamma' = d\gamma\left(\frac{\partial}{\partial t}\right) \text{ is vecta field along } \gamma$$



$$\begin{tikzcd}
 & TM \arrow[d] \\
{[a,b]} \arrow[r, "\gamma"'] \arrow[ur, bend left, "d\gamma\left(\frac{\partial}{\partial t}\right)"] \arrow[ur, phantom, "\circlearrowleft" near end] & M
\end{tikzcd}$$

We define $D_{\gamma'}\gamma' \overset{def}{=\!=} \tilde{D}_{\frac{\partial}{\partial t}}\gamma'$.

(Check: If γ is embedded, this definition coincides with the previous one.)

$\therefore$ Geodesic (& P^{γ}) can be defined for any smooth curve.

Ch3 Covariant derivative, Curvature Tensor

3.1 Covariant derivative of tensors

Fact: Let $\varphi : V \to W$ be an isomorphism between vector spaces, then φ can be extended to an isomorphism between the tensor algebras:

$$\tilde{\varphi} : \bigoplus_{r,s} T^{r,s} V \to \bigoplus_{r,s} T^{r,s} W,$$

where $T^{r,s} V = (\underbrace{V \otimes \cdots \otimes V}_{r}) \otimes (\underbrace{V^* \otimes \cdots \otimes V^*}_{s})$,

V^* = dual of V.

In fact, we can first define

$$\begin{array}{ccc} \varphi^*: W^* & \longrightarrow & V^* \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \alpha & \longmapsto & \varphi^*(\alpha) \end{array} \quad \text{by} \quad \boxed{\varphi^*(\alpha)(v) = \alpha(\varphi(v))}$$

Then φ = isom $\Rightarrow$ φ^* isom

i.e. $(\varphi^*)^{-1} : V^* \to W^*$ exists

Hence we can define

$\forall \; v_1 \otimes \cdots \otimes v_r \otimes \alpha^1 \otimes \cdots \otimes \alpha^s \in T^{r,s} V$,

$$\tilde{\varphi}(v_1 \otimes \cdots \otimes v_r \otimes \alpha^1 \otimes \cdots \otimes \alpha^s)$$

$$= \varphi(v_1) \otimes \cdots \otimes \varphi(v_r) \otimes (\varphi^*)^{-1}(\alpha^1) \otimes \cdots \otimes (\varphi^*)^{-1}(\alpha^s) \in T^{r,s} W.$$

Finally, extend $\tilde{\varphi}$ to all $\bigoplus_{r,s} T^{r,s} V$ by linearity and can be checked that $\tilde{\varphi}$ is an isomorphism.

<u>Def</u>: Let M = Riemannian manifold, $x \in M$, $v \in T_x M$,

γ = curve with $\gamma(0) = x$, $\gamma'(0) = v$.

Then $\forall$ tensor field K on M, we define the <u>covariant derivative</u> of K <u>wrt v</u> by

$$D_v K = \frac{d}{dt}\Big|_{t=0} \tilde{P}_t^{-1}\left(K(\gamma(t)) \right)$$

where $\tilde{P}_t : \bigoplus_{r,s} T^{r,s}(T_x M) \longrightarrow \bigoplus_{r,s} T^{r,s}(T_{\gamma(t)} M)$

is the extension of the parallel transport

$P_t : T_x M \longrightarrow T_{\gamma(t)} M$ wrt Levi-Civita connection.

Caution: We need to check $D_v K$ does not depend on γ.

Properties:

(1) If K is a (r,s)-tensor, then $D_v K$ is also a (r,s)-tensor.

(2) D_v is a derivation on the tensor algebra:

$$D_v (K_1 \otimes K_2) = (D_v K_1) \otimes K_2 + K_1 \otimes (D_v K_2)$$

(3) D_v commutes with "contractions".

Def (of contraction) The contractions C_{pq}, $p = 1, \cdots, r$, $q = 1, \cdots, s$

are linear maps

$$C_{pq} : \left(\bigotimes^r TM\right) \otimes \left(\bigotimes^s T^*M\right) \to \left(\bigotimes^{r-1} TM\right) \otimes \left(\bigotimes^{s-1} T^*M\right)$$

defined by

$$C_{pq}(v_1 \otimes \cdots \otimes v_r \otimes \alpha^1 \otimes \cdots \otimes \alpha^s)$$

$$= \alpha^q(v_p)\; v_1 \otimes \cdots \otimes \hat{v}_p \otimes \cdots \otimes v_r \otimes \alpha^1 \otimes \cdots \otimes \widehat{\alpha^q} \otimes \cdots \otimes \alpha^s$$

omitted

egs: For $C_{11} : TM \otimes T^*M \to \mathbb{R} \; (\simeq \bigotimes^0 TM \otimes \bigotimes^0 T^*M)$

takes $\frac{\partial}{\partial x^i} \otimes dx^j \longmapsto C_{11}\left(\frac{\partial}{\partial x^i} \otimes dx^j\right) = dx^j\left(\frac{\partial}{\partial x^i}\right) = \delta^j_i$

For $C_{11} : TM \otimes \bigotimes^2 T^*M \longrightarrow T^*M$

takes $\frac{\partial}{\partial x^i} \otimes dx^{j_1} \otimes dx^{j_2} \mapsto C_{11}\left(\frac{\partial}{\partial x^i} \otimes dx^{j_1} \otimes dx^{j_2}\right)$

$$= dx^{j_1}\left(\frac{\partial}{\partial x^i}\right) dx^{j_2} = \delta^{j}_{i}\, dx^{j_2} \in T^*M$$

Property (3) means if $\mathcal{C} = C_{pq}$ is a contraction, then

$$\boxed{D_v(\mathcal{C}K) = \mathcal{C}(D_v K)}.$$

Pf: (1) is clear.

(2) We do a special case only. The general case can be proved similarly.

Suppose $K = X \otimes \omega \otimes \rho \in TM \otimes (\otimes^2 T^*M)$

i.e. X = vector field,

ω, ρ = 1-forms (ie linear combinations of dx^i)

Then we need to prove that

$$D_v K = (D_v X) \otimes \omega \otimes \rho + X \otimes D_v \omega \otimes \rho + X \otimes \omega \otimes D_v \rho$$

Let $\{e_1(t), \cdots, e_n(t)\}$ be parallel vector fields along γ

s.t. $\{e_i(t)\}$ forms a basis of $T_{\gamma(t)}M$.

i.e. $D_{\gamma'} e_i(t) \equiv 0$.

Then $\forall t$, $\exists$ dual basis $\{\alpha^1(t), \cdots, \alpha^n(t)\}$ of $T^*_{\gamma(t)}M$,

i.e. $\alpha^i(t)(e_j(t)) = \delta^i_j$, $\forall t$.

By definition of $\tilde{P}_t$, we see that

$$\tilde{P}_t(\alpha^i(0)) \overset{\text{def}}{=} (P_t^*)^{-1}(\alpha^i(0))$$

$$\Longleftrightarrow \quad P_t^*(\tilde{P}_t(\alpha^i(0))) = \alpha^i(0)$$

$$\Longleftrightarrow \quad P_t^*(\tilde{P}_t(\alpha^i(0)))(e_j(0)) = \alpha^i(0)(e_j(0)) = \delta^i_j \quad \forall j$$

$$\Longleftrightarrow \quad \tilde{P}_t(\alpha^i(0))(P_t(e_j(0))) = \delta^i_j \quad \forall j$$

$$\Longleftrightarrow \quad \tilde{P}_t(\alpha^i(0))(e_j(t)) = \delta^i_j \quad \forall j$$

$$\Longleftrightarrow \quad \tilde{P}_t(\alpha^i(0)) = \alpha^i(t)$$

i.e. $\{\alpha^i(t)\}$ are all parallel.

Write
$$\begin{cases} X(t) = X(\gamma(t)) = \sum_i X^i(t) e_i(t) \\ \omega(t) = \omega(\gamma(t)) = \sum_j \omega_j(t) \alpha^j(t) \\ \rho(t) = \rho(\gamma(t)) = \sum_l \rho_l(t) \alpha^l(t) \end{cases}$$

Then $K(t) = \sum_{i,j,l} X^i(t) \omega_j(t) \rho_l(t) \; e_i(t) \otimes \alpha^j(t) \otimes \alpha^l(t)$

$$\Rightarrow \tilde{P}_t^{-1} K(t) = \sum_{i,j,l} X^i(t) \omega_j(t) \rho_l(t) \; e_i(0) \otimes \alpha^j(0) \otimes \alpha^l(0)$$

$$\Rightarrow D_v K = \frac{d}{dt}\Big|_{t=0} \tilde{P}_t^{-1} K(t)$$

$$= \sum_{i,j,l} \left(\frac{dX^i}{dt} \omega_j \rho_l + X^i \frac{d\omega_j}{dt} \rho_l + X^i \omega_j \frac{d\rho}{dt} \right) e_i(0) \otimes \alpha^j(0) \otimes \alpha^l(0) .$$

Similarly
$$\begin{cases} D_v X = \sum_i \dfrac{dX^i}{dt} e_i(0) \\ D_v \omega = \sum_j \dfrac{d\omega_j}{dt} \alpha^j(0) \\ D_v \rho = \sum_l \dfrac{d\rho_l}{dt} \alpha^l(0) \end{cases}$$

$$\Rightarrow \quad D_v K = D_v X \otimes \omega \otimes \rho + X \otimes D_v \omega \otimes \rho + X \otimes \omega \otimes D_v \rho$$

This proves (2).

Pf of (3) We do the special case that
$$K = X \otimes \omega \otimes \rho \in TM \otimes (\otimes^2 T^*M) \quad \&$$
$$\tau = C_{12} : TM \otimes (\otimes^2 T^*M) \to T^*M$$

In this case $\quad \tau K = \tau(X \otimes \omega \otimes \rho)$

$$= \rho(X)\,\omega \in T^*M$$

$$\mathcal{C}(D_v K) = \mathcal{C}\left(D_v X \otimes \omega \otimes \rho + X \otimes D_v \omega \otimes \rho + X \otimes \omega \otimes D_v \rho\right)$$

$$= \rho(D_v X)\,\omega + \rho(X) D_v \omega + (D_v \rho)(X)\,\omega$$

$\therefore$ We need to show that

$$D_v(\rho(X)\omega) = \rho(D_v X)\,\omega + \rho(X) D_v \omega + (D_v \rho)(X)\,\omega .$$

Note that
$$\rho(X) = \left(\sum_\ell \rho_\ell \alpha^\ell(t)\right)\left(\sum_i X^i e_i(t)\right)$$
$$= \sum_{\ell, i} \rho_\ell X^i \delta_i^\ell = \sum_i \rho_i X^i$$

$$\rho(D_v X) = \sum_i \rho_i \frac{dX^i}{dt}$$

$$(D_v \rho)(X) = \sum_i \frac{d\rho_i}{dt} X^i$$

$$\Rightarrow \quad \rho(D_v X)\,\omega + \rho(X) D_v \omega + (D_v \rho)(X)\,\omega$$

$$= \left[\left(\rho_i \frac{dX^i}{dt}\right) \omega_j + (\rho_i X_i) \frac{d\omega_j}{dt} + \left(\frac{d\rho_i}{dt} X^i\right) \omega_j \right] \alpha^j(0)$$

and $D_v(\rho(X)\omega) = D_v\left((\rho_i X^i)\, \omega_j\, \alpha^j(t) \right)$

$$= \frac{d}{dt}\Big|_{t=0} \left[\left[(\rho_i X^i)\omega_j \right] \alpha^j(0) \right]$$

$$= \rho(D_v X)\,\omega + \rho(X) D_v \omega + (D_v \rho)(X)\,\omega$$

\#

Note: • One can define $D_v \rho$ by

$$D_v\left[\mathcal{C}(X \otimes \rho) \right] = \mathcal{C}\left(D_v (X \otimes \rho) \right)$$

i.e. $v(\rho(X)) = \mathcal{C}(D_v X \otimes \rho + X \otimes D_v \rho)$

$= \rho(D_v X) + (D_v \rho)(X)$

i.e. $\boxed{(D_v \rho)(X) = v(\rho(X)) - \rho(D_v X) \quad \forall X \in \Gamma(TM)}$

- This also shows that $D_v K$ does not depend on γ (since the RHS does not depend on γ).

Def: Let K = tensor field on M,

X = vector field on M

Then we define $(D_X K)(x) \overset{\text{def}}{=} D_{X(x)} K$, $\forall x \in M$.

Note: By linearity of $D_X K$ in X, on can define

$$DK \in \left(\bigotimes^r TM\right) \otimes \left(\bigotimes^{s+1} T^*M\right) \quad \left(\text{for } K \in \left(\bigotimes^r TM\right) \otimes \left(\bigotimes^s T^*M\right)\right)$$

by requiring

$$DK(\omega^1 \otimes \cdots \otimes \omega^r \otimes X_1 \otimes \cdots \otimes X_s \otimes X)$$

$$\underline{\underline{\text{def}}} \ (D_X K)(\omega^1 \otimes \cdots \otimes \omega^r \otimes X_1 \otimes \cdots \otimes X_s)$$

[**Caution:** Some authors put

$$DK(\omega^1 \otimes \cdots \otimes \omega^r \otimes X \otimes X_1 \otimes \cdots \otimes X_s) = (D_X K)(\cdots)$$]

Note: If $K = f \in T^{(0,0)}M \cong C^\infty(M)$.

Then $Df = df$ the usual differential of f.

Def: For $n \geq 0$, we define

$$D^{n+1} K = D(D^n K)$$

Note: $D^2 K(\cdots, X, Y) \neq D_Y(D_X K)(\cdots)$ in general.

eg: Let $K = f \in C^\infty(M)$

Then
$$\begin{aligned} D^2 f(X, Y) &= \big(D(df)\big)(X, Y) \\ &= \big(D_Y(df)\big)(X) \\ &= Y(df(X)) - df(D_Y X) \\ &= YXf - (D_Y X)f \\ &\neq D_Y(D_X f) \end{aligned}$$

(by definition $D_Y(D_X f) = D_Y(Xf) = Y(Xf) = YXf$)

<u>Note</u>: $D^2 f\,(X,Y) = YXf - (D_Y X)(f)$

$D^2 f\,(Y,X) = XYf - (D_X Y)(f)$

$$\Rightarrow \quad D^2 f(X,Y) - D^2 f(Y,X) = -[X,Y]f + (D_X Y - D_Y X) f$$

$$= \underset{\uparrow}{T}(X,Y) f$$

torsion tensor

$\therefore$ D symmetric (torsion free) $\iff$ $D^2 f$ is symmetric

In this case, $D^2 f$ is called the <u>Hessian</u> of f.

From now on, we assume M has a Riemannian metric g, and D = Levi-Civita connection of g.

Therefore $D^2 f$ is always symmetric $\forall f \in C^\infty(M)$.

Def: $\forall S \in \otimes^2 T^*M$, we define $\operatorname{tr} S \in C^\infty(M)$ the trace of S, by

$$\operatorname{tr} S(x) = \sum_i S(e_i, e_i)$$

where $\{e_i\}$ is an orthonormal basis of $T_x M$.

Note: • $\operatorname{tr} S$ is well-defined, i.e. independent of the choice of o.n. basis $\{e_i\}$.

• $(\operatorname{tr} S)(x)$ is smooth in x

(Pf: Ex)

<u>Def</u>: Let (M, g) = Riemannian manifold

D = Levi-Civita connection of g.

Then the <u>Laplace operator</u>, <u>Laplacian</u> or <u>Laplace-Beltrami operator</u>

$$\Delta : C^\infty(M) \longrightarrow C^\infty(M)$$

is defined by $\Delta f = \operatorname{tr} D^2 f$.

<u>Ex</u>: Prove that in local coordinates $(x^1, \cdots, x^n)$

$$\Delta f = \frac{1}{\sqrt{G}} \sum_j \frac{\partial}{\partial x^j}\left(\sum_i g^{ij}\sqrt{G}\,\frac{\partial f}{\partial x^i} \right)$$

where $G = \det(g_{ij})$, $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ & $(g^{ij}) = (g_{ij})^{-1}$