Ch2 Riemannian Metric, Connection & Parallel Transport.

Ref: 伍鴻熙，沈純理，虞言林 "黎曼几何初步"，北京大學出版社

2.1 Riemannian metric & connection

Def: Let M be a C^∞ manifold. A Riemannian metric g on M is given by an inner product g_x on each T_xM which depends smoothly on $x \in M$ in the sense that in any coordinates system U with coordinate functions $x^1, \cdots, x^n$,

$$g_{ij}(x) = g_x\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \qquad (\forall i, j)$$

is a smooth function on the nbd.

Notation, most of the time we write

$$\langle \, , \, \rangle_x \quad \text{for } g_x$$

$$(\text{and } \langle \, , \, \rangle \quad \text{for } g.)$$

<u>Note</u>: • By definition, $(g_{ij}(x))$ is a <u>symmetric</u> <u>positive definite</u> $n \times n$ matrix $\forall x \in U$.

• g can be regarded as a $(0,2)$-tensor satisfying

$$\begin{cases} g(X, X) \geq 0 & \forall X \in \Gamma(TM) \\ g_x(X, X) = 0 \Leftrightarrow X(x) = 0 \\ g(X, Y) = g(Y, X), & \forall X, Y \in \Gamma(TM) \end{cases}$$

Hence $\boxed{g = \sum_{i,j=1}^{n} g_{ij}(x) \, dx^i \otimes dx^j}$ in local coordinates

Def: A <u>connection</u> D (∇) on a C^∞ manifold M is

a mapping $D : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$

$$(V, X) \longmapsto D_V X,$$

such that

(C1) $D_{fV+gW} X = f D_V X + g D_W X$

(C2) $D_V(fX) = (Vf) X + f D_V X$

(C3) $D_V(X+Y) = D_V X + D_V Y$

where $V, W, X, Y \in \Gamma(TM)$; $f, g \in C^\infty(M)$.

(and $Vf = D_V f$ is the directional derivative of f in direction V)

Note: $D_V X$ is called the covariant derivative of X in the direction of V.

Fact: If $V, W \in \Gamma(TM)$ are vector fields s.t. $V(x) = W(x)$, then $(D_V X)(x) = (D_W X)(x)$, $\forall X \in \Gamma(TM)$.

(Pf: Ex.)

Using this fact, we have

Def: $\forall v \in T_x M$, one can define

$$D_v X \overset{\text{def}}{=} (D_V X)(x) \quad (\in T_x M)$$

where V is a vector field s.t. $V(x) = v$.

eg: Standard connection on $\mathbb{R}^n$

Recall the direction derivative of function

$$D_v f = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t|v|}$$

for a smooth function defined near $x \in \mathbb{R}^n$.

A smooth vector field X on $\mathbb{R}^n$ can be written as

$$X = \sum X^i(x) \frac{\partial}{\partial x^i}$$

(x^i = standard coordinates on $\mathbb{R}^n$, $\frac{\partial}{\partial x^i} = e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th}$)

where $X^i(x)$ are smooth functions

Then $D_v X \overset{\text{def}}{=\!=} \sum D_v X^i(x) \frac{\partial}{\partial x^i}$, and

$$(D_V X)(x) \overset{\text{def}}{=\!=} D_{V(x)} X$$

define a connection on $\mathbb{R}^n$ (check: C1 – C3)

(By definition, we must have $D_v\left(\frac{\partial}{\partial x_j}\right)=0$, $\forall j=1,\cdots,n$)

<u>Lemma</u>: The set of connections on M is <u>convex</u>.

i.e. If $D^1,\cdots,D^k$ are connections on M

$f_1,\cdots,f_k$ are functions $\in C^\infty(M)$ with

$$\sum_{i=1}^k f_i = 1,$$

then $D=\sum_{i=1}^k f_i D^i$ is a connection on M.

$$\left(D_v X \overset{\text{def}}{=} \sum f_i D^i_v X \right)$$

Pf: C1 & C3 are clear & do not need $\sum f_i = 1$.

For C2, we have

$$D_V(f X) = \sum_i f_i D^i_V(f X)$$

$$= \sum_i f_i \left[(Vf) X + f D^i_V X \right]$$

$$= (Vf) X + f D_V X \quad \left(\text{since } \sum_i f_i = 1\right)$$

※

Thm Let M be a C^∞ manifold. Then $\exists$ a connection on M.

Pf: Let $\{(U_i, \phi_i)\}$ be an atlas on M

Then $\{U_i\}$ is an open cover of M

$\Rightarrow$ $\exists$ partitions of unity $\{\varphi_i\}$ subordinate to $\{U_i\}$

(WLOG, we may assume $\{V_k\}_{k\in\Lambda'} = \{U_i\}_{i\in\Lambda}$)

On each U_i, the standard connection on $\mathbb{R}^n$ induces a connection $\overset{\circ}{D}{}^i$. Then $\sum \varphi_i D^i$ is a connection on M by the previous lemma. $\divideontimes$

<u>Remark</u>: Similar argument shows that there exists Riemannian metric on any manifold.

<u>Lemma</u>: Let $v \in T_xM$, and $\gamma : [0,\varepsilon) \to M$ be a curve such that $\gamma'(0) = v$. Suppose $X, Y \in \Gamma(TM)$

be 2 vector fields s.t. $X(\gamma(t)) = Y(\gamma(t))$, $\forall t \in [0, \varepsilon)$

Then $D_v X = D_v Y$.

(i.e. $D_{\gamma'(0)} X$ is determined by $X \circ \gamma$)

(Pf: Ex)

<u>Thm</u>: Let M = manifold

$g = \langle \, , \, \rangle$ = Riemannian metric on M

Then $\exists !$ connection D s.t.

(compatible with g) (L1) $X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle$

(torsion free) (L2) $D_X Y - D_Y X - [X, Y] = 0$.

Pf: (Uniqueness)

In coordinates, any vector field can be written as

$$X = \sum X^i \frac{\partial}{\partial x^i}$$

$$\Rightarrow \qquad D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k} \quad \text{for some } \Gamma_{ij}^k \text{ (functions)}$$

Now for $X = X^j \frac{\partial}{\partial x^j}$, $V = V^i \frac{\partial}{\partial x^i}$, then

$$D_V X = D_{V^i \frac{\partial}{\partial x^i}} \left(X^j \frac{\partial}{\partial x^j} \right) = V^i D_{\frac{\partial}{\partial x^i}} \left(X^j \frac{\partial}{\partial x^j} \right)$$

$$= V^i \left(\frac{\partial X^j}{\partial x^i} \frac{\partial}{\partial x^j} + X^j D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right)$$

$$= v^i \left(\frac{\partial X^j}{\partial x^i} \frac{\partial}{\partial x^j} + X^j \Gamma_{ij}^k \frac{\partial}{\partial x^k} \right)$$

$$= v^i \left(\frac{\partial X^k}{\partial x^i} + \Gamma_{ij}^k X^j \right) \frac{\partial}{\partial x^k}$$

$\therefore \{\Gamma_{ij}^k\}$ determines $D_v X$.

Let $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle \ \forall i, j$

$$\Rightarrow \frac{\partial}{\partial x^i} g_{jk} = \frac{\partial}{\partial x^i} \langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \rangle$$

$$= \langle D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \rangle + \langle \frac{\partial}{\partial x^j}, D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} \rangle$$

$$= \langle \Gamma_{ij}^l \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k} \rangle + \langle \frac{\partial}{\partial x^j}, \Gamma_{ik}^l \frac{\partial}{\partial x^l} \rangle$$

$$= g_{lk}\Gamma^{l}_{ij} + g_{jl}\Gamma^{l}_{ik}$$

$$\Rightarrow \begin{cases} \dfrac{\partial g_{jk}}{\partial x^i} = g_{lk}\Gamma^{l}_{ij} + g_{jl}\Gamma^{l}_{ik} & \text{—— (1)} \\[2ex] \dfrac{\partial g_{ki}}{\partial x^j} = g_{li}\Gamma^{l}_{jk} + g_{kl}\Gamma^{l}_{ji} & \text{—— (2)} \\[2ex] \dfrac{\partial g_{ij}}{\partial x^k} = g_{lj}\Gamma^{l}_{ki} + g_{il}\Gamma^{l}_{kj} & \text{—— (3)} \end{cases}$$

Note that by (L2),

$$0 = D_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} - D_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i} - \cancelto{0}{\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]}$$

$$= \left(\Gamma^{k}_{ij} - \Gamma^{k}_{ji}\right)\frac{\partial}{\partial x^k}$$

$$\Rightarrow \quad \boxed{\Gamma_{ij}^{k} = \Gamma_{ji}^{k}} \quad \forall i, j, k$$

Then (1)+(2)−(3) $\Rightarrow$

$$\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} = 2 g_{lk} \Gamma_{ij}^{l}$$

Denote the inverse matrix of (g_{ij}) by (g^{ij}).

Then $\quad g^{sk} g_{kl} = \delta_l^s \quad \forall s, l$

$$\Rightarrow \quad \boxed{\Gamma_{ij}^{k} = \frac{1}{2} g^{kl} \left[\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right]} \quad \text{— } (\Gamma)$$

$\therefore \{\Gamma_{ij}^{k}\}$ & hence D satisfying L1 & L2 is uniquely determined by g.

<u>Existence</u>: Let $\{(U_\beta, \phi_\beta)\}$ = atlas of M. For $X = X^j \frac{\partial}{\partial x^j}$

& $V = V^i \frac{\partial}{\partial x^i}$ on U_β, we define

$$D_V X \overset{\text{def}}{=} V^i \left(\frac{\partial X^k}{\partial x^i} + \Gamma_{ij}^k X^j \right) \frac{\partial}{\partial x^k}$$

with Γ_{ij}^k defined by (Γ)

Then one can check that $D_V X$ doesn't depend on the coordinate (U_β, ϕ_β). Hence it defines a connection D on M. The properties L1 & L2 are then easy to check. $\divideontimes$

Note:

- The connection given by this theorem is called the Levi-Civita connection of g, (a Riemannian connection of g)
- The coefficients Γ_{ij}^{k} of D are called Christoffel symbols if D is Levi-Civita.
- The formula (Γ) is equivalent to

$$\langle D_X Y, Z\rangle = \frac{1}{2}\Big\{ X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle + \langle Z,[X,Y]\rangle + \langle Y,[Z,X]\rangle - \langle X,[Y,Z]\rangle \Big\}$$

for $X, Y, Z \in \Gamma(TM)$

eg On S^3, there exist $\hat{i}, \hat{j}, \hat{k}$ orthonormal vector fields such that $[\hat{i}, \hat{j}] = \hat{k}$, $[\hat{j}, \hat{k}] = \hat{i}$ & $[\hat{k}, \hat{i}] = \hat{j}$.

$$\begin{aligned}\langle D_{\hat{i}}\hat{j}, \hat{k}\rangle &= \frac{1}{2}\Big\{ \hat{i}\cancelto{0}{\langle \hat{j}, \hat{k}\rangle} + \hat{j}\cancelto{0}{\langle \hat{k}, \hat{i}\rangle} - \hat{k}\cancelto{0}{\langle \hat{i}, \hat{j}\rangle} \\ &\qquad + \langle \hat{k}, [\hat{i}, \hat{j}]\rangle + \langle \hat{j}, [\hat{k}, \hat{i}]\rangle - \langle \hat{i}, [\hat{j}, \hat{k}]\rangle \Big\} \\ &= \frac{1}{2}\Big\{ \langle \hat{k}, \hat{k}\rangle + \langle \hat{j}, \hat{j}\rangle - \langle \hat{i}, \hat{i}\rangle \Big\} = \frac{1}{2}\end{aligned}$$

Similarly, $\langle D_{\hat{i}}\hat{j}, \hat{i}\rangle = \langle D_{\hat{i}}\hat{j}, \hat{j}\rangle = 0$

Hence $D_{\hat{i}}\hat{j} = \frac{1}{2}\hat{k}$ (Similarly for others: Ex.)

Geometry meaning of Levi-Civita connection

<u>Def</u>: Let N be a (embedded) submanifold of M. Suppose g is a metric on M, then the <u>induce metric</u> $\bar{g}$ of g on N is defined by

$$\bar{g}(X,Y) = g(X,Y), \quad \forall X,Y \in TN \subset TM$$

(eg. If $N \subset M$ is open, then $\bar{g} = g|_N$)

<u>Def</u>: Let (M,g) be a Riemannian manifold, D = Levi-Civita connection of g.

Suppose $N \subset M$ is a submanifold, then one can define a connection on N by

$$\bar{D}_{\underline{X}} Y = (D_{\underline{X}} Y)^{\perp}$$

where $(\ )_x^{\perp} : T_x M \to T_x N$ is the <u>orthogonal</u> <u>projection</u> (wrt g_x on $T_x M$)

<u>Facts</u>
- $\bar{D}$ is well-defined, ie. $\bar{D}$ satisfies C1 – C3.
- $\bar{D}$ is the Levi-Civita connection of the induce metric $\bar{g}$. (Pf: Ex)

<u>Note</u>: If $M = \mathbb{R}^n$, g = standard metric (= flat metric) then Levi-Civita connection D = usual directional derivative.

Hence, the facts above give a geometry interpretation of the Levi-Civita connection on submanifolds N of $\mathbb{R}^n$.

"Meaning" of L2: $D_X Y - D_Y X - [X, Y] = 0$

L2 doesn't use the metric g, and in local coordinates

$$L2 \Leftrightarrow \Gamma_{ij}^k = \Gamma_{ji}^k$$

Hence, connections satisfying (L2) are called symmetric

Moreover, $T(X, Y) = D_X Y - D_Y X - [X, Y]$

defines a $(1,2)$-tensor on M called the torsion tensor,

i.e. $T \in \Gamma(TM \otimes (\otimes^2 T^*M))$ (i.e. linear in X, Y (Ex))

Hence D is symmetric $\Leftrightarrow T \equiv 0$

$\Leftrightarrow D$ is torsion free.

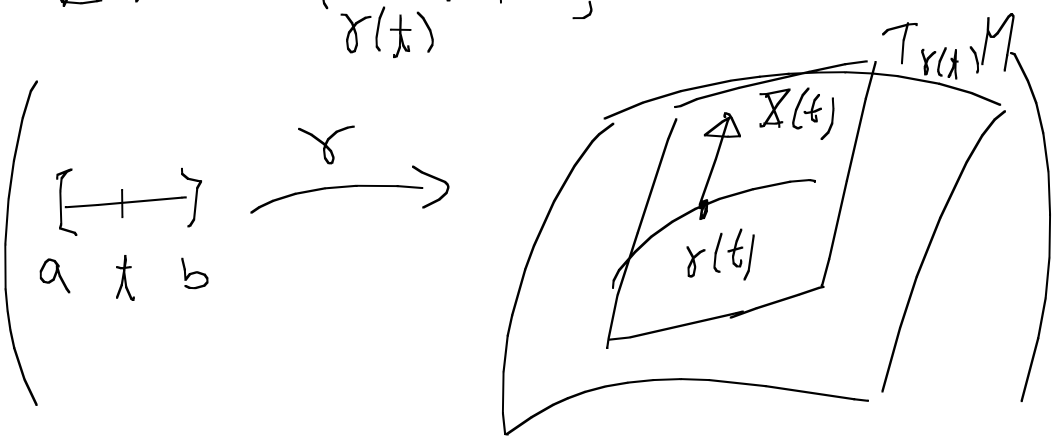
2.2 Parallel Transport

Let D be a connection (not necessary Levi-Civita) on M;

$\gamma : [a,b] \to M$ be an embedded curve such that $\gamma([a,b])$ is contained in a coordinate neighborhood with coordinate functions $\{x^i\}$.

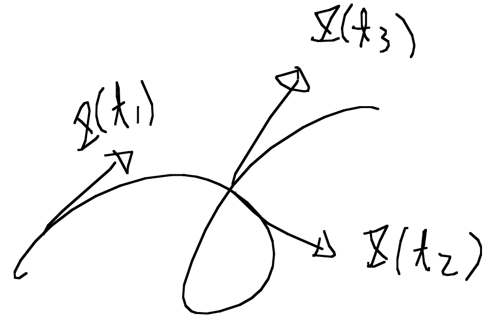
Suppose X is a vector field along γ, i.e., X depends smoothly on t and $X(t) \in T_{\gamma(t)}M$, $\forall t \in [a,b]$

$$\begin{tikzcd} & TM \arrow[d] \\ {[a,b]} \arrow[ur, "X"] \arrow[r, "\gamma"'] \arrow[ur, phantom, "\circlearrowleft", shift right=3ex] & M \end{tikzcd}$$



Since γ is embedded, X can be extended to a smooth vector field $\tilde{X}$ on M.

(Not true for immerse curve :



)

Now for any 2 extensions $\tilde{X}$ & $\tilde{Y}$, we must have

$$\tilde{X}(\gamma(t)) = \tilde{Y}(\gamma(t)) = X(\gamma(t))$$

$$\Rightarrow \quad D_{\gamma'(t)} \tilde{X} = D_{\gamma'(t)} \tilde{Y}$$

$\therefore$ $\underline{D_{\gamma'(t)} X \text{ is well-defined}}$.

In local coordinates,

$$\gamma'(t) = \sum \gamma'^{i}(t) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)}$$

$$Z(t) = \sum Z^i(t) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)}$$

for some functions $\gamma'^i(t)$ & $Z^i(t)$.

Recall that
$$D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x^k} \quad \left(\text{for some } \Gamma^k_{ij}\right)$$

Therefore
$$D_{\gamma'(t)} Z = D_{\gamma'(t)} \left(Z^j \frac{\partial}{\partial x^j}\right)$$
$$= \left(D_{\gamma'(t)} Z^j\right) \frac{\partial}{\partial x^j} + Z^j D_{\gamma'(t)} \frac{\partial}{\partial x^j}$$
$$= \frac{dZ^j}{dt} \frac{\partial}{\partial x^j} + Z^j \gamma'^i D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}$$
$$= \left(\frac{dZ^k}{dt} + Z^j \gamma'^i \Gamma^k_{ij}\right) \frac{\partial}{\partial x^k}$$

$$D_{\gamma'(t)} X = 0 \iff \frac{dX^k}{dt} + (\Gamma^k_{ij} \gamma'^i) X^j = 0 \ , \ \forall k = 1, \cdots, n$$

linear ODE system in $X^1, \cdots, X^n$.

Linear ODE theory $\Rightarrow$

$\forall v \in T_{\gamma(a)} M$, $\exists !$ soln. $X(t)$ to the IVP

$$\begin{cases} D_{\gamma'(t)} X = 0 \ , & \forall \ t \in [a, b] \\ X(a) = v \end{cases}$$

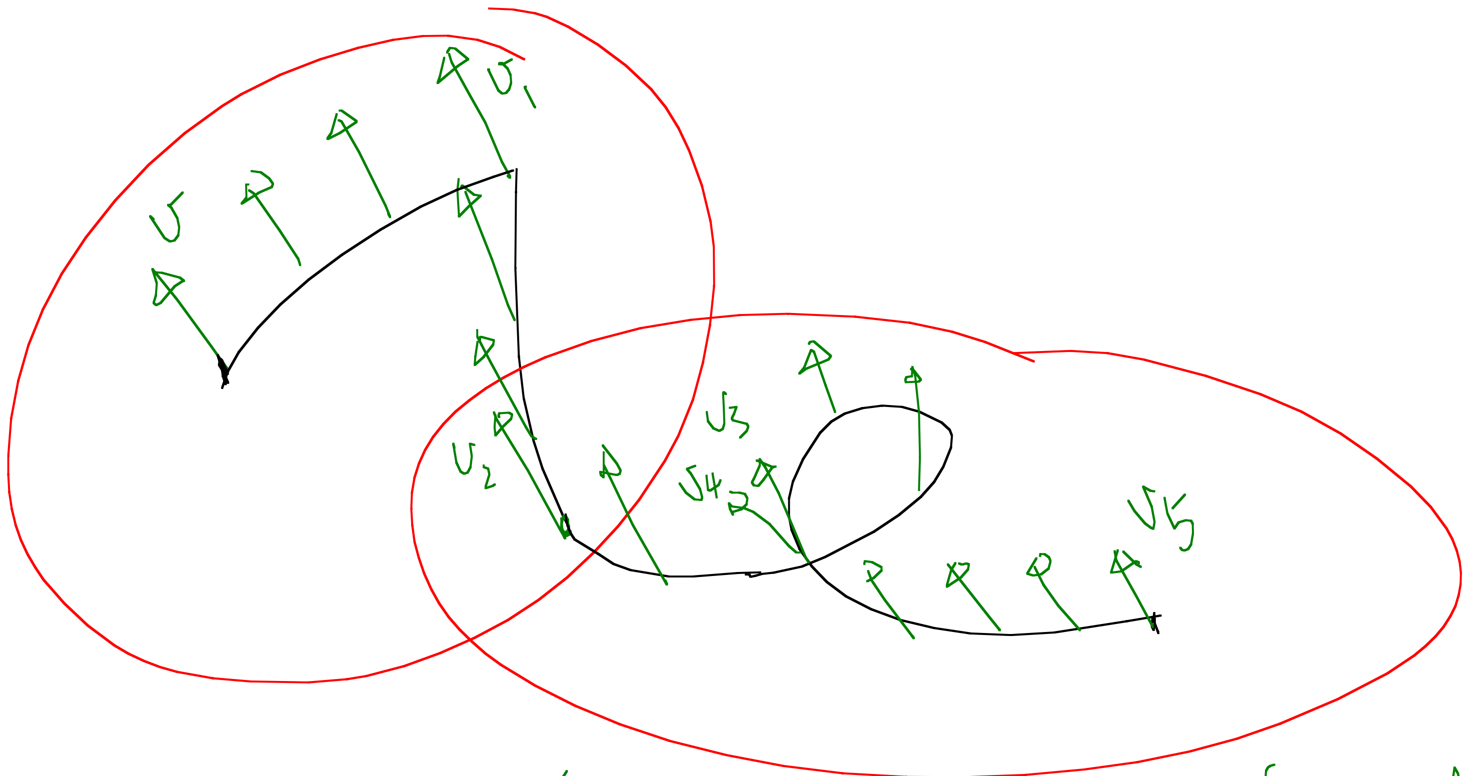
Def: A vector field X along γ is called parallel if $D_{\gamma'} X = 0$.

Def: A vector $w \in T_{\gamma(b)} M$ is called a parallel transport of a vector $v \in T_{\gamma(a)}M$ along γ if $\exists$ a parallel vector field X along γ such that

$$X(a) = v \quad \& \quad X(b) = w$$

Note: parallel transport w of v (along γ) is uniquely determined by v. (for fixed γ)

Note: If γ is not embedded or contained in a chart or γ is only piecewise smooth, we can use subdivision to define parallel transport of a vector $v \in T_{\gamma(a)}M$ along γ.



(V_3 may not equal to V_4 for curved space)

Hence we have

Thm $\forall$ immersed curve $\gamma : [a,b] \longrightarrow M$ & $v \in T_{\gamma(a)} M$, $\exists !$ parallel vector field $X(t)$ along γ s.t. $X(a) = v$.

Hence $\exists !$ $w \in T_{\gamma(b)} M$ s.t. w is the parallel transport of v along γ.

This Thm $\Rightarrow$ one can define $\forall$ immersed curve $\gamma : [a,b] \to M$

a mapping

$$P^\gamma : T_{\gamma(a)}M \longrightarrow T_{\gamma(b)}M$$

$$\begin{array}{ccc} \Downarrow & & \Downarrow \\ v & \longmapsto & w = \text{parallel transport of } v \text{ along } \gamma. \end{array}$$

<u>Thm</u>: $P^\gamma : T_{\gamma(a)}M \to T_{\gamma(b)}M$ is an vector space isomorphism.

(Pf : Ex.)

- P^γ is called <u>parallel transport</u> from <u>$\gamma(a)$ to $\gamma(b)$ along γ</u>.

Furthermore, if D is the Levi-Civita connection of a metric g on M, then $\forall$ 2 parallel vector fields X & Y along γ (γ embedded)

$$\frac{d}{dt}\langle X, Y\rangle = \gamma'(t)\langle X, Y\rangle$$

$$= \langle \cancelto{0}{D_{\gamma'(t)} X}, Y\rangle + \langle X, \cancelto{0}{D_{\gamma'(t)} Y}\rangle$$

$$= 0$$

$\therefore$ $P^{\gamma} : T_{\gamma(a)}M \to T_{\gamma(b)}M$ is in fact an isometry of the inner product spaces.

Conversely, if D is a connection such that all P^{γ} are isometries of the inner product spaces, then $\forall$ vector fields X, Y, Z, we choose a curve $\gamma : [0,1] \to M$

s.t. $\gamma'(0) = X(x)$ $(x \in M)$

Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of T_xM. Then parallel transport P^γ along γ defines orthonormal basis $\{e_1(t), \cdots, e_n(t)\}$ of $T_{\gamma(t)}M$, $\forall t \in [0,1]$ (since P^γ are isometries $\forall t$)

Hence
$$Y(\gamma(t)) = \sum Y^i(t)\, e_i(t)$$
$$Z(\gamma(t)) = \sum Z^i(t)\, e_i(t)$$
for some $Y^i(t)$ & $Z^i(t)$

$$\Rightarrow \quad X(x)\langle Y, Z\rangle = \gamma'(0)\langle Y, Z\rangle$$
$$= \frac{d}{dt}\Big|_{t=0} \langle Y, Z\rangle(\gamma(t))$$

$$= \frac{d}{dt}\Big|_{t=0} Y^i(t) Z^i(t)$$

$$= \frac{dY^i}{dt}(0)\, Z^i(0) + Y^i(0)\, \frac{dZ^i}{dt}(0)$$

Note that

$$D_{\gamma'(0)} Y = D_{\gamma'(0)} \left(\sum Y^i(t)\, e_i(t) \right)$$

$$= \sum \frac{dY^i}{dt}(0)\, e_i + \sum Y^i(0)\, \cancel{D_{\gamma'(0)} e_i}^{\to 0}$$

$$= \sum \frac{dY^i}{dt}(0)\, e_i$$

Similarly for $D_{\gamma'(0)} Z$.

$\Rightarrow \quad X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle$.

$\Rightarrow \quad D$ is compatible with the metric g.

Conclusion: D = compatible with $g \Leftrightarrow P^\gamma$ = isometry, $\forall \gamma$.

In particular, if D is symmetric,

$$D = \text{Levi-Civita} \Leftrightarrow P^\gamma = \text{isometry}, \forall \gamma.$$

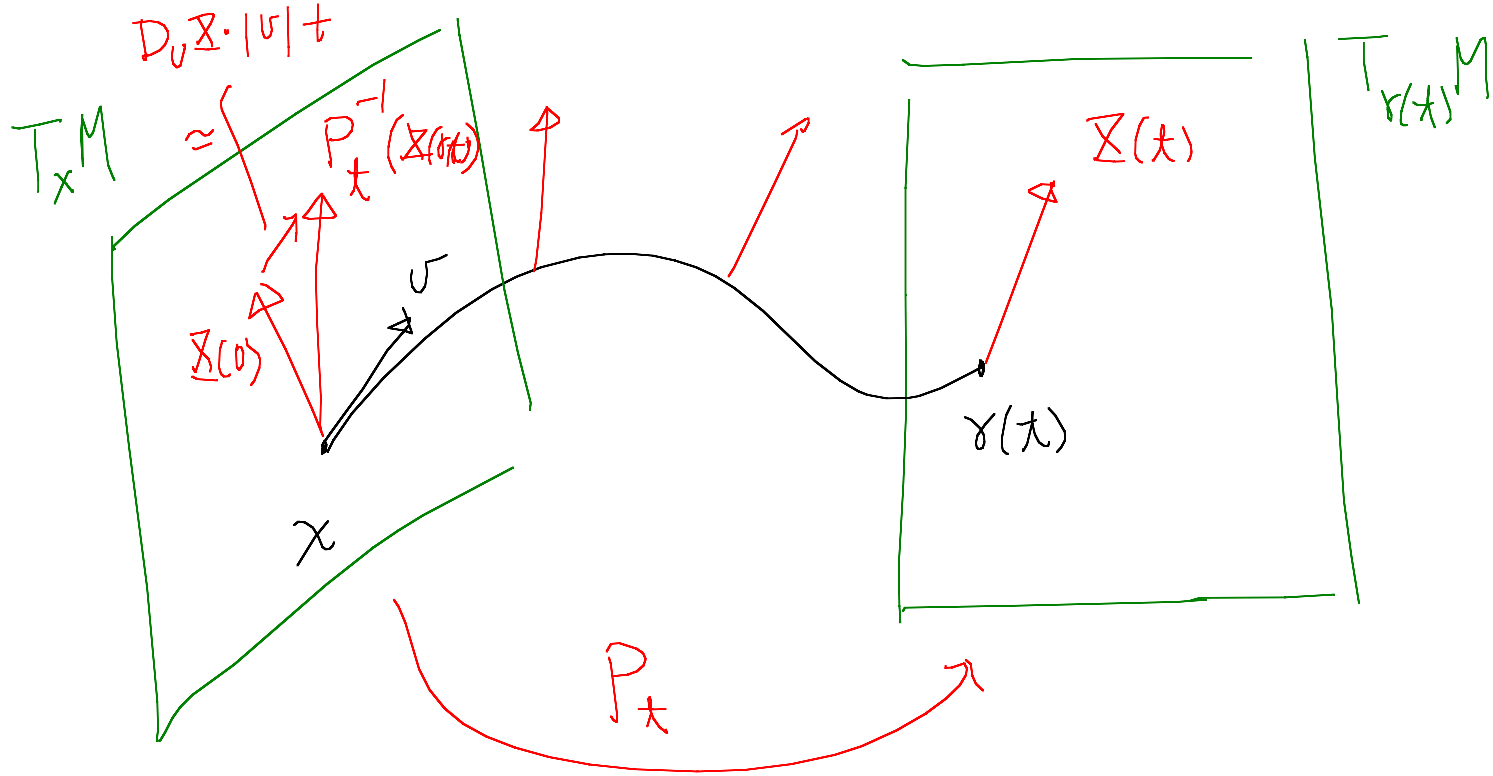
Thm: $\forall v \in T_xM$ & $X \in \Gamma(TM)$,

$$D_v X = \left.\frac{d}{dt}\right|_{t=0} P_t^{-1}\big(X(\gamma(t))\big) \quad \left(\begin{array}{c}\text{for } D\\ \text{Levi-Civita}\end{array}\right)$$

where $\gamma : [0,1] \to M$ is a curve s.t.

$\gamma(0) = x$ & $\gamma'(0) = v$

$P_t : T_xM \to T_{\gamma(t)}M$ = parallel transport along $\gamma|_{[0,t]}$.



Pf: Let $\{e_i\}$ be an orthonormal basis of $T_x M$.

Define $e_i(t) = P_t e_i$

Then $\{e_i(t)\}$ is an o.n. basis of $T_{\gamma(t)} M$.

Write X in terms of $\{e_i(t)\}$:

$$X(\gamma(t)) = \sum \hat{X}^i(t)\, e_i(t) \quad \text{for some } \hat{X}^i(t)$$

$$\Rightarrow \quad D_v X = \sum \frac{d\hat{X}^i}{dt}(0)\, e_i$$

And $$P_t^{-1}(X(\gamma(t))) = \sum \hat{X}^i(t)\, P_t^{-1}(e_i(t))$$

$$= \sum \hat{X}^i(t)\, e_i \quad \in T_x M$$

$$\Rightarrow \quad \frac{d}{dt}\Big|_{t=0} P_t^{-1}(X(\gamma(t))) = \sum \frac{d\hat{X}^i}{dt}(0)\, e_i = D_v X.$$

$\#$