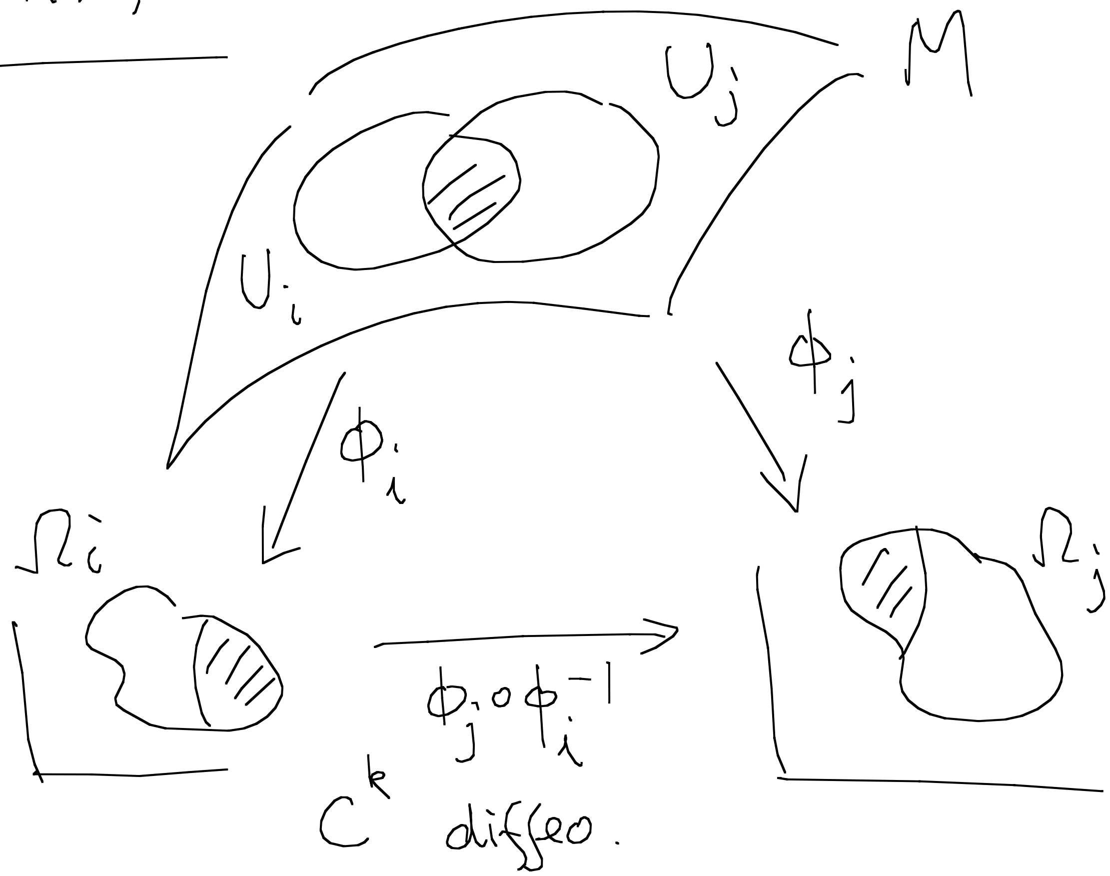
Ch 1 Differentiable Manifolds

1.1 Abstract Manifolds



Def: A C^k atlas on a Hausdroff topological space M is given by

(i) an open covering U_i, $i \in \Lambda$, of M;

(ii) a family of homeomorphisms

$$\phi_i : U_i \to \Omega_i \subset \mathbb{R}^n \quad (\Omega_i \text{ is open})$$

such that $\forall\ i, j \in \Lambda$

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

is a C^k diffeomorphism.

Remark:

- $\phi_j \circ \phi_i^{-1}$, $i, j \in \Lambda$ (with $U_i \cap U_j \neq \emptyset$) are called transition functions.
- (U_i, ϕ_i) is a called a (coordinate) chart,
- $\phi_i^{-1} : \Omega_i \to U_i \subset M$ is a local parametrization.

Def: Two C^k atlas for M, say $(U_i, \phi_i)_{i\in\Lambda_1}$ and $(V_j, \psi_j)_{j\in\Lambda_2}$, are <u>$C^k$ equivalent</u> if <u>their union</u> is still a C^k atlas, that is, if $\forall i \in \Lambda_1$, $j \in \Lambda_2$ (st $U_i \cap V_j \neq \emptyset$)

$$\phi_i \circ \psi_j^{-1} : \psi_j(U_i \cap V_j) \to \phi_i(U_i \cap V_j)$$

are C^k diffeomorphisms.

Def: A <u>differentiable structure</u> of <u>class</u> C^k on M is an equivalence class of C^k atlas.

<u>Remark</u>: If M is connected, then the integer n in the definition does not depend on the chart and is defined as the <u>dimension</u> of M.

Def: A C^k differentiable manifold of dimension n is a pair $(M, \mathcal{A})$, where M is a Hausdorff top. space and $\mathcal{A} = \{(U_i, \phi_i)\}_{i \in \Lambda}$ is a C^k atlas on M with $\phi_i(U_i) \subset \mathbb{R}^n$.

Remark:
- In this course, we consider only C^∞ differentiable manifold which is connected and a further condition such that "partitions of unity" is always possible.
- All compact manifolds satisfy the further condition.
- We'll refer such a manifold as a smooth manifold (or even simply manifold.)

eg: $M = T^n$, the n-torus $\left(T^n = \underbrace{S^1 \times \cdots \times S^1}_{n}\right)$

Let $f : \mathbb{R}^n \to T^n \subset \mathbb{C}^n$

$\qquad\quad \rotatebox{90}{$\in$} \qquad\quad \rotatebox{90}{$\in$}$ $\qquad\qquad$ (f is onto)

$(x_1, \cdots, x_n) \mapsto (e^{ix_1}, \cdots, e^{ix_n})$

$\forall p \in T^n$, $\exists\ x^p = (x_1^p, \cdots, x_n^p) \in \mathbb{R}^n$ s.t.

$p = f(x^p)$ $\left(\text{one may choose } x_i^p \in [0, 2\pi),\ i = 1, \cdots, n\right)$

Consider $\Omega_p = (x_1^p - \pi, x_1^p + \pi) \times \cdots \times (x_n^p - \pi, x_n^p + \pi) \subset \mathbb{R}^n$

and let $\begin{cases} U_p = f(\Omega_p) \subset T^n & (U_p \text{ open \& contains } p) \\ \phi_p = (f|_{\Omega_p})^{-1} : U_p \to \Omega_p \subset \mathbb{R}^n & \text{homeo.} \end{cases}$

Then $\{(U_p, \phi_p)\}_{p \in T^n}$ is an C^∞ altas on T^n:

In fact, if $p, q \in T^n$ s.t. $U_p \cap U_q \neq \emptyset$,

then $\phi_q \circ \phi_p^{-1}(x_1, \cdots, x_n)$ $\quad \left((x_1, \cdots, x_n) \in \phi_p(U_p \cap U_q) \subset \Omega_p \right)$

$$= \phi_q(f(x_1, \cdots, x_n))$$

$$= \phi_q(e^{ix_1}, \cdots, e^{ix_n}) \qquad \left((e^{ix_1}, \cdots, e^{ix_n}) \in U_p \cap U_q \right)$$

$$= (f|\Omega_q)^{-1}(e^{ix_1}, \cdots, e^{ix_n})$$

$$= (x_1 + 2k_1\pi, \cdots, x_n + 2k_n\pi) \quad \text{for some } k_1, \cdots, k_n$$

$$\text{s.t. } x_i + 2k_i\pi \in (x_i^q - \pi, x_i^q + \pi)$$

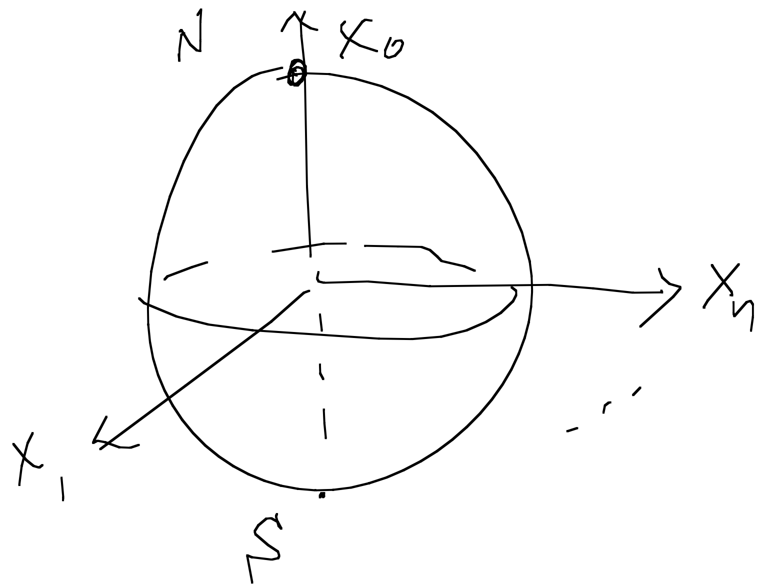
note that k_i are indep. of $(x_1, \cdots, x_n) \in \phi_p(U_p \cap U_q)$

hence $\phi_q \circ \phi_p^{-1}$ is just a translation in $\mathbb{R}^n$.

Therefore $\phi_q \circ \phi_p^{-1}$ is a C^∞ diffeo.

$\Rightarrow$ $(T^n, \{(U_p, \phi_p)\}_{p\in T^n})$ is a smooth manifold.

<u>eg</u> $M = S^n$, the n-sphere $S^n = \{(x_0, x_1, \dots, x_n) : \sum_{j=0}^{n} x_j^2 = 1\} \subset \mathbb{R}^{n+1}$



Let $\begin{cases} N = (1, 0, \dots, 0) \in S^n \\ S = (-1, 0, \dots, 0) \in S^n \end{cases}$

$\begin{cases} U_1 = S^n \setminus \{N\} \\ U_2 = S^n \setminus \{S\} \end{cases}$

$U_1 \cup U_2 = S^n$

Let

$$\begin{cases} \phi_1 : U_1 \to \mathbb{R}^n & \text{(Stereographic projections)} \\ \quad \rotatebox{90}{$\in$} & \\ (x_0, x_1, \cdots, x_n) \longmapsto \dfrac{1}{1-x_0}(x_1, \cdots, x_n) & \\ & \\ \phi_2 : U_2 \to \mathbb{R}^n & \\ \quad \rotatebox{90}{$\in$} & \\ (x_0, x_1, \cdots, x_n) \longmapsto \dfrac{1}{1+x_0}(x_1, \cdots, x_n) & \end{cases}$$

are homeomorphisms.

Note that if $\phi_1(x_0, x_1, \cdots, x_n) = (y_1, \cdots, y_n)$

then

$$\phi_1^{-1}(y_1, \cdots, y_n) = \left(\frac{|y|^2 - 1}{|y|^2+1}, \frac{2y_1}{|y|^2+1}, \cdots, \frac{2y_n}{|y|^2+1} \right)$$

If $y \neq 0$

$$\phi_2 \circ \phi_1^{-1}(y_1, \cdots, y_n) = \frac{1}{1 + \frac{|y|^2 - 1}{|y|^2+1}} \left(\frac{2y_1}{|y|^2+1}, \cdots, \frac{2y_n}{|y|^2+1} \right)$$

$$= \frac{1}{|y|^2}(y_1, \cdots, y_n)$$

In short

$$\boxed{\phi_2 \circ \phi_1^{-1}(y) = \frac{y}{|y|^2} \quad \forall y \in \mathbb{R}^n \setminus \{0\}}$$

which is a C^∞ diffeomorphism

$\Rightarrow \mathcal{A} = \{(U_1, \phi_1), (U_2, \phi_2)\}$ is an altas on S^n,

therefore $(S^n, \mathcal{A})$ is a smooth manifold.

<u>eg</u> $\mathbb{RP}^n$ the <u>real projective space</u> (in some book: $P^n\mathbb{R}$)

- As topological space

$\mathbb{RP}^n$ = quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ by the equivalent relation:

$$x \sim y \iff \exists \lambda \neq 0 \in \mathbb{R} \text{ s.t. } x = \lambda y$$

$(x, y \in \mathbb{R}^{n+1} \setminus \{0\})$

$$= S^n / \{\pm Id\}$$

(Hence $\mathbb{RP}^n$ is Hausdroff, compact, connected.)

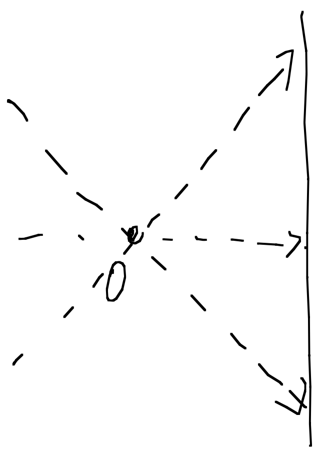
- Let $\pi : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{RP}^n$ be the canonical projection map

i.e. $\pi(x)$ = equi. class of x.

Define $V_i = \{ x = (x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} : x_i \neq 0 \}$

$$\Phi_i : V_i \longrightarrow \mathbb{R}^n$$
$$x \longmapsto \left(\frac{x_0}{x_i}, \cdots, \widehat{\frac{x_i}{x_i}}, \cdots, \frac{x_n}{x_i} \right)$$

this means the term deleted.



$\{x_i = 1\} \cong \mathbb{R}^n$

Then $\forall\, x, y \in V_i$, we have

$(*)\quad \Phi_i(x) = \Phi_i(y) \iff \pi(x) = \pi(y)$ (ie. $x \sim y$)

(check!)

This gives

$$\begin{tikzcd} \mathbb{R}^{n+1} \supset V_i \arrow[r, "\Phi_i"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{R}^n \\ U_i = \pi(V_i) \arrow[ur, "\phi_i"'] & \end{tikzcd}$$

where ϕ_i is defined by

$$\phi_i : U_i = \pi(V_i) \longrightarrow \mathbb{R}^n$$
$$\text{eq. class of } x \longmapsto \Phi_i(x)$$

(ϕ_i is well-defined because of $(*)$).

Using π: $\quad \phi_i(\pi(x)) = \Phi_i(x) \quad \sim \quad \phi_i \circ \pi = \Phi_i$

Further $\phi_i : U_i \to \mathbb{R}^n$ is homeomorphism (Check)
with inverse

$$\phi_i^{-1}(y_0, \cdots, y_{n-1}) = \pi\left(y_0, \cdots, y_{i-1}, 1, y_i, \cdots, y_{n-1}\right)$$

Therefore, if $y_j \neq 0$, for $j < i$, we have

$$(\phi_j \circ \phi_i^{-1})(y_0, \cdots, y_{n-1}) = \phi_j\left(\pi(y_0, \cdots, y_{i-1}, 1, y_i, \cdots, y_{n-1})\right)$$

$$= \Phi_j(y_0, \cdots y_j, \cdots, 1, y_i, \cdots, y_{n-1})$$

$$= \left(\frac{y_0}{y_j}, \cdots, \frac{\hat{y_j}}{y_j}, \cdots, \frac{1}{y_j}, \frac{y_i}{y_j}, \cdots, \frac{y_{n-1}}{y_j}\right)$$

$\therefore$ $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a C^∞ diffeo,

Hence $\mathbb{R}P^n$ with the atlas $\{(U_i, \phi_i)\}_{i=0}^n$ is a smooth manifold.

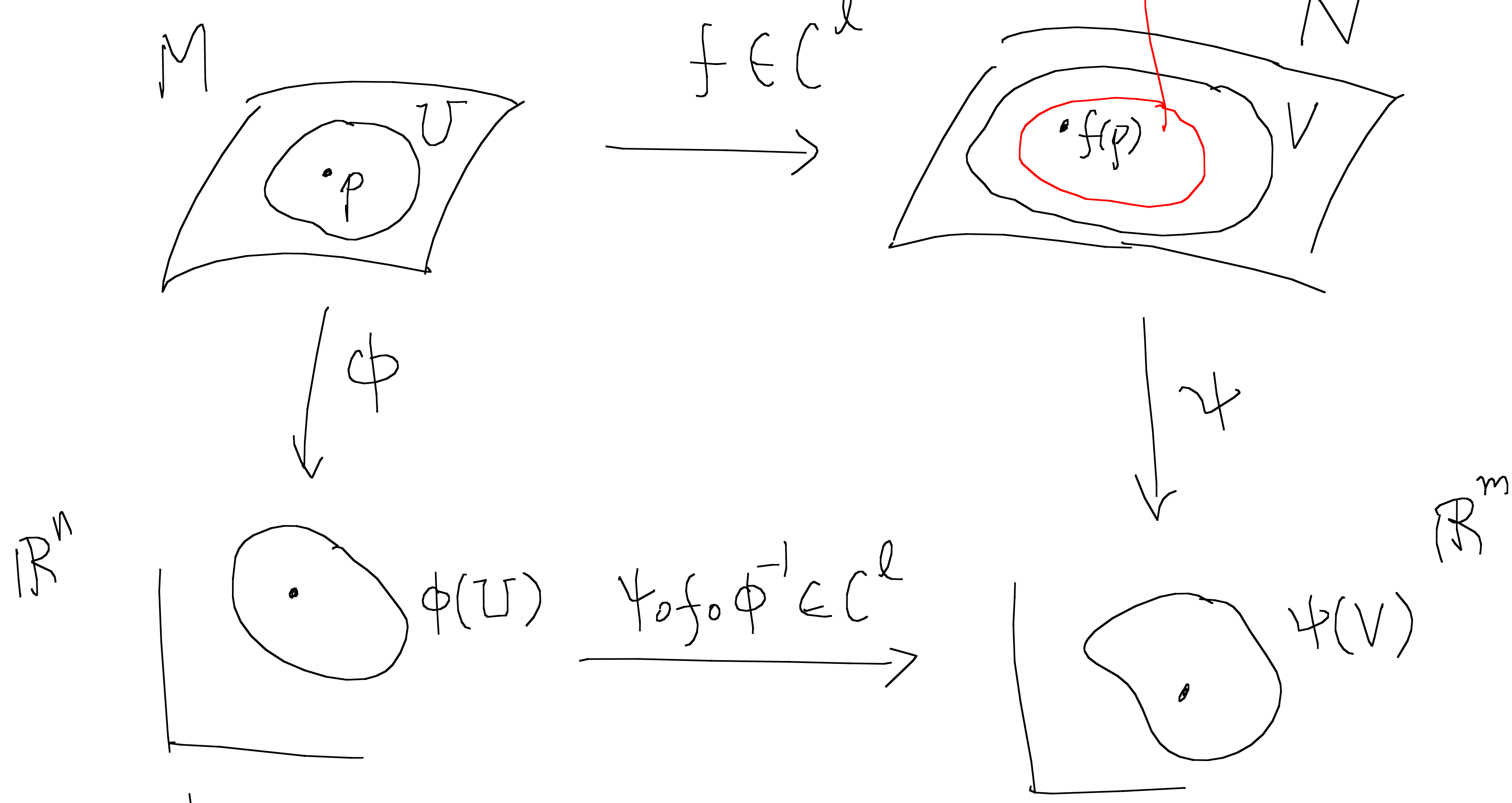
Note: $\mathbb{RP}^n$ is <u>non-orientable</u> for n <u>even</u> according to the following definition: (proof omitted)

<u>Def</u>: A smooth manifold M is said to be <u>orientable</u> if $\exists$ an altas on M s.t.

$$J(\phi_j \circ \phi_i^{-1}) > 0 \quad \forall\, i, j$$

Jacobian determinant of $\phi_j \circ \phi_i^{-1}$.

1.2 Smooth Maps



($\psi \circ f \circ \phi^{-1}$ is just m functions of n variables as in Calculus)

Def: Let M & N be C^k manifolds. A continuous map $f: M \to N$ is $\underline{C^\ell}$ map (for $\ell \leq k$) if $\forall p \in M$, $\exists$ charts (U, ϕ) & (V, ψ) for M & N around p & $f(p)$ respectively with $f(U) \subset V$ such that

$$\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V) \text{ is } C^\ell$$

Note: This definition does not depend on the charts since transition functions are C^k ($k \geq \ell$)

Def: A C^k map $\gamma : (a,b) \to M$ from an interval to a smooth manifold is called a $\underline{C^k \text{ curve}}$ (on M).

Def: A C^k map $f: M \to \mathbb{R}$ (or $\mathbb{C}$) is called a <u>C^k function</u> on M.

Def: A smooth map $g: \mathbb{R}^n \to \mathbb{R}^k$ is a <u>submersion</u> (an <u>immersion</u>, a <u>local diffeomorphism</u>) at $x \in \mathbb{R}^n$ if $D_x g : \mathbb{R}^n \to \mathbb{R}^k$ is <u>surjective</u> (<u>injective</u>, bijective)

Def: Let M & N be smooth manifolds. A smooth map $f: M \to N$ is a <u>submersion</u> (<u>immersion</u>, <u>local diffeomorphism</u>) at $p \in M$, if $\exists$ charts (U, ϕ) for M around p, (V, ψ) for N around $f(p)$

with $f(U) \subset V$ s.t. $\psi \circ f \circ \phi^{-1}$ is a submersion

(immersion, local diffeomorphism) at $\phi(p) \in \phi(U) \subset \mathbb{R}^n$.

Def: A map $f: M \to N$ is a submersion (immersion, local diffeomorphism)

if it has the property at any point of M.

Def: A map $f: M \to N$ is a diffeomorphism if it is a

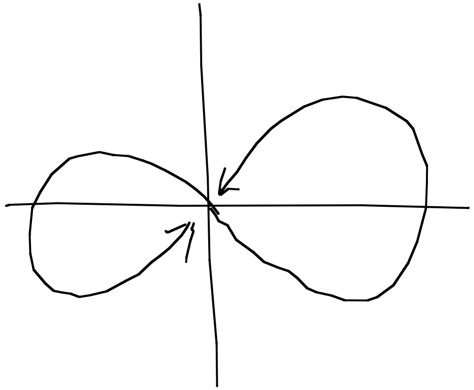
bijection such that both f and f^{-1} are smooth.

Def: A map $f: M \to N$ is an embedding if it is an

immersion and $f: M \to f(M) \subset N$ (with subspace top)

is a homeomorphism.

eg: $\mathbb{R} \xrightarrow{\gamma}$



$|\gamma'| \neq 0$

this is an immersion but not embedding (Ex)

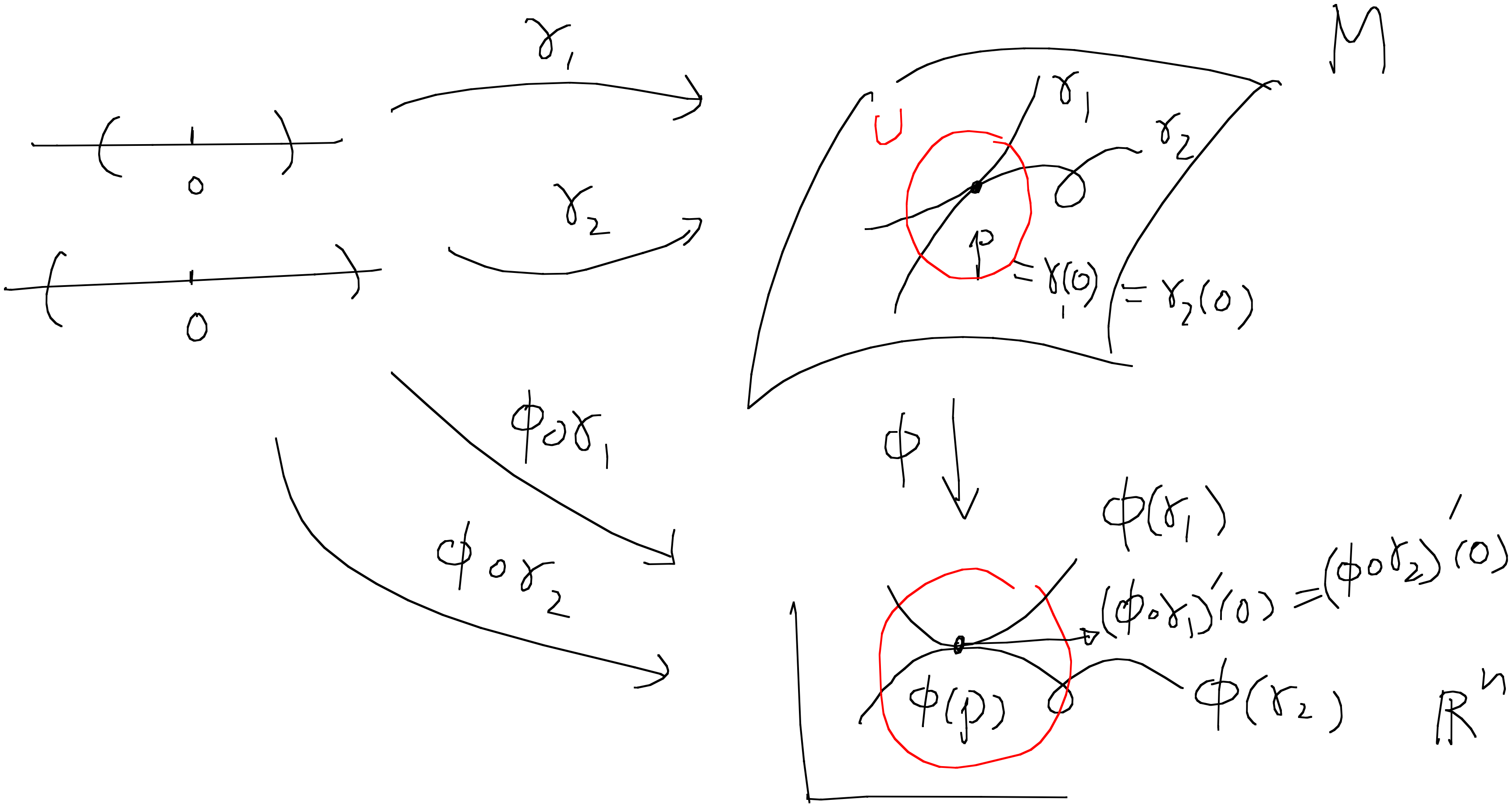
1.3 Tangent vectors

Def 1: Let M be a smooth manifold and $p \in M$.

A tangent vector to M at p is an equi. class of C^∞ curves $\gamma : I \to M$, where I = interval containing 0, such that $\gamma(0) = p$, for the equi. relation defined by $\gamma_1 \sim \gamma_2 \iff$

$$(\phi\circ\gamma_1)'(0) = (\phi\circ\gamma_2)'(0)$$

for a chart (U,ϕ) around p.



Ex: Check that the equi. relation is well-defined by showing that for any other chart (V,ψ) around p, we have

$$(\psi\circ\gamma)'(0) = D_{\phi(p)}(\psi\circ\phi^{-1})\cdot(\phi\circ\gamma)'(0)$$

where $D_{\phi(p)}(\psi\circ\phi^{-1})$ is the Jacobi matrix (a differential) of the map $\psi\circ\phi^{-1}$ at $\phi(p)$.

Def 2 (Equivalent definition for tangent vectors)

Let M be a smooth manifold, $p\in M$. (U,ϕ) & (V,ψ) be 2 coordinates charts for M around p. Let u, v be 2 vectors in $\mathbb{R}^n$ (considered as tangent vectors to $\mathbb{R}^n$ at $\phi(p)$ & $\psi(p)$ respectively) We say that

$$\boxed{(U,\phi,u)\simeq(V,\psi,v) \iff D_{\phi(p)}(\psi\circ\phi^{-1})u = v}$$

Then a tangent vector to M at p is a equi. class of triples (U, ϕ, u).

<u>Note</u>: • In def 1, a tangent vector is represented by a curve γ. We usually write $\gamma'(0)$ for the tangent vector $[\gamma]$ for simplicity (Independent of charts)

• In def 2, the "same" tangent vector will be represented in a chart (U, ϕ) by a vector $u \in \mathbb{R}^n$.

• Def 1 $\Leftrightarrow$ Def 2 by taking $u = (\phi \circ \gamma)'(0)$.

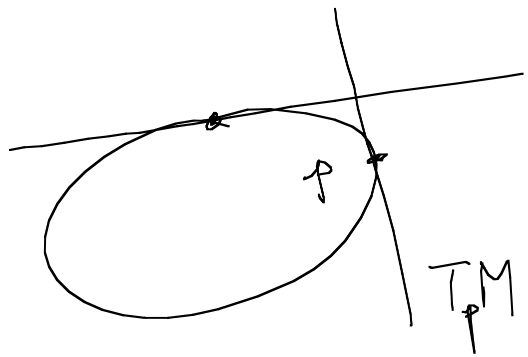
<u>Notation</u>: The set of tangent vectors to M at p is denoted by T_pM. (Tangent space to M at $p \in M$).

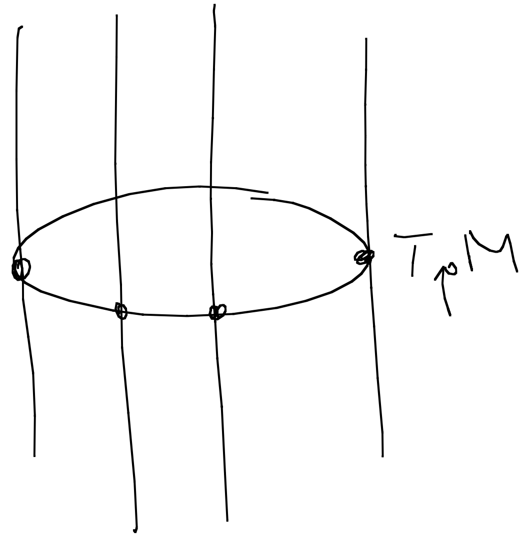
__Note:__ If a chart (U, ϕ) is given, then we have an "isomorphism"

$$\begin{array}{rcl} \theta_{U,\phi,p} : \mathbb{R}^n & \longrightarrow & T_pM \\ \cup\!\!\!\!\!\!\; & & \cup\!\!\!\!\!\!\; \\ u & \longmapsto & [(U, \phi, u)] \end{array}$$

(check: 1-1, onto)

__Def:__ The <u>disjoint</u> union TM of T_pM, $\forall p \in M$, is called the <u>tangent bundle</u> of M.



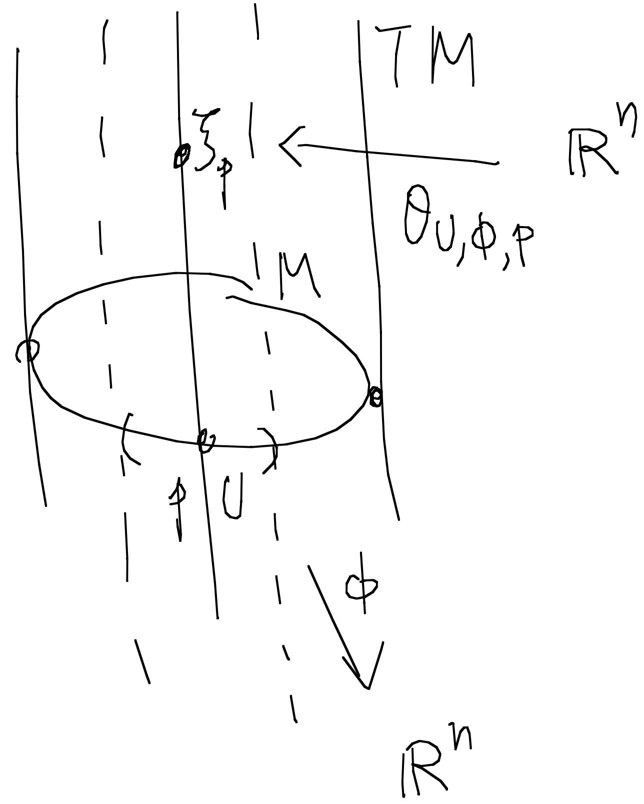


<u>Thm</u>: Let M be an n-dim. C^p manifold ($p>1$). Then TM can be equipped with a <u>$2n$</u>-dim'l <u>C^{p-1}</u> abstract manifold structure.

<u>Pf</u>: (Sketch)

For each chart (U,ϕ) of M, define a "chart"

$$\left(\coprod_{p\in U} T_pM,\ \Phi\right) \text{ for } TM$$



by

$$\Phi(\xi_p) = (\phi(p), \theta^{-1}_{U,\phi,p}(\xi_p)) \in \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$$

$\forall\, \xi_p \in T_pM$, $p \in U$

Then one can see all these "$\coprod_{p\in U} T_pM$" give a topology on TM such that Φ are homeomorphisms. And one can check that TM is Hausdroff and

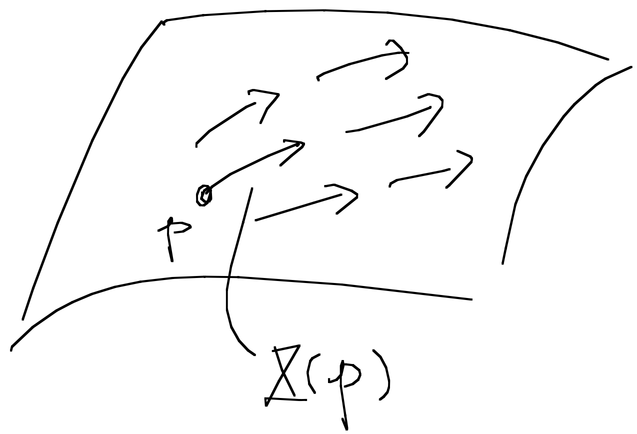
$\{(\coprod_{p\in U} T_pM, \Phi)\}_{(U,\phi)}$ forms an C^{p-1} altas of TM.

(we've differentiated once in the equi. relation for tangent vectors)

<u>Def</u>: A (<u>smooth</u>) <u>vector field</u> X on a manifold M is a smooth <u>section</u> of the <u>tangent bundle</u> TM of M,

i.e. $X : M \to TM$ is a smooth map

s.t. $X(p) \in T_pM$



- The set of vector fields on M is denoted by $\Gamma(TM)$.