4.3 Completeness, metric structure.

(M, g) = Riemannian manifold (connected)

Def: $d : M \times M \to [0, \infty)$ defined by

$$d(x, y) = \inf_{\gamma} L(\gamma),$$

where "inf" is taken over all piecewise smooth curves γ joining x and y, is called the distance (a metric) of (M, g).

Thm: (M, d) is a metric space, i.e. d satisfies

(1) $d(x,y) \geq 0$; "=" iff $x=y$.

(2) $d(x,y) = d(y,x)$,

(3) $d(x,y) \leq d(x,z) + d(z,y)$.

Pf: All are easy (Exs.) and we prove only

"$d(x,y)=0 \Rightarrow x=y$".

Suppose $x \neq y$. If $y \in B_\delta$, where δ is given as in the "Thm" in the previous section, then

$d(x,y) = L(\gamma)$, where γ = radial geodesic from x to y.

$\Rightarrow d(x,y) > 0$,

Continuity argument $\Rightarrow$ $d(x,y) = \delta > 0$ if $y \in \partial B_\delta$.

Hence if $y \notin B_\delta$, and σ = curve joining x to y.

Choose the 1st point y_1 of σ on ∂B_σ and conclude that

$$L(\sigma) \geq L\left(\sigma \big|_{(\text{from } x \text{ to } y_1)}\right) \geq \delta > 0$$

Taking "inf" $\Rightarrow$ $d(x,y) \geq \delta > 0$ ⌘

In fact, we have a stronger theorem

<u>Thm</u>: The topology of (M, d) is the same as the

original topology of M.

(Pf: Ex or pages 61-62 of H. Wu or do Carmo.)

<u>Def</u>: A Riemmann manifold (M, g) is said to be <u>complete</u> if the associated metric space (M, d) is complete.

<u>egs</u>: $(\mathbb{R}^n$, standard metric$)$, $(S^n$, standard metric$)$ are complete

<u>Hopf-Rinow Thm</u>: The following statements are equivalent on a Riemannian manifold (M, g):

(1) M is complete;

(2) $\forall\, x \in M$, $\exp_x$ defined on the whole T_xM;

(3) $\exists\, x \in M$, $\exp_x$ defined on the whole T_xM;

(4) bounded closed subsets of M are compact.

Cor 1 of Hopf-Rinow Thm

If (M, g) is complete, then $\forall\, x \neq y \in M$,

$\exists$ a minimizing geodesic γ joining x and y.

(Recall: all manifolds discussed in this course are assumed to be connected.)

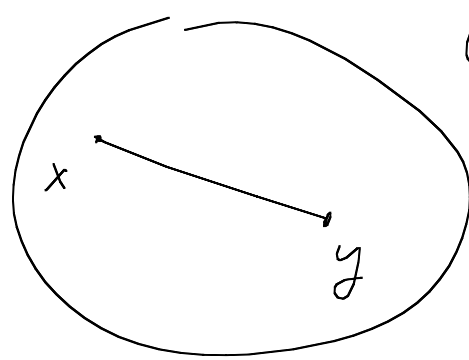
Cor 2 : If (M, g) is complete, then $\forall x \in M$,

$$\exp_x : T_x M \to M \quad \text{is surjective.}$$

Notes :

- The converse of Cor 1 of Hopf-Rinow Thm is not true in general :

eg



convex in $\mathbb{R}^n$

incomplete

- A general complete metric space may <u>not</u> have Heine-Borel property ((4) of the thm) eg: $S = \{a_1, a_2, \dots\}$ countable infinite set of distinct elements.

Define discrete metric d on S by

$$d(a_i, a_j) = 1 - \delta_{ij}$$

Then (S, d) is a complete metric space which is bounded.

$\Rightarrow$ S is a closed & bounded set but not compact.

Pf of Hopf-Rinow Thm:

(1) $\Rightarrow$ (2) Let $\gamma : [0, \delta) \to M$ be a geodesic

$\gamma(t) = \exp_x(tv)$ for some $v \in T_xM$.

Suppose that $I = (a_1, b_1)$ is the maximal possible interval containing $[0, \delta)$ s.t. $\gamma(t)$ is defined.

Suppose $b_1 < +\infty$. Then M complete $\Rightarrow$

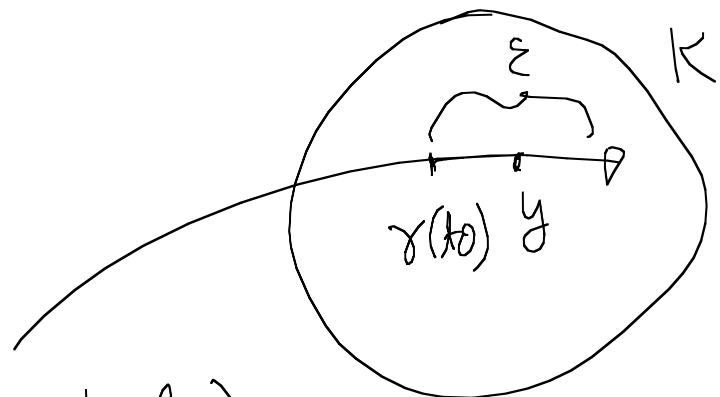
$\exists\, y \in M$ s.t. $\lim_{t \to b_1} \gamma(t) = y$.

Let K = cpt nbd. of y.

ODE theory $\Rightarrow$ $\exists\, \varepsilon > 0$ indep. of t_0 s.t.

" If $d(\gamma(t_0), y) < \frac{\varepsilon}{2}$, Then $\exists$ geodesic $\zeta : [0, \varepsilon] \to M$ s.t.

$$\zeta(0) = \gamma(t_0) \quad \& \quad \zeta'(0) = \gamma'(t_0) \text{ "}.$$



(Ex: check the detail)

$\Rightarrow$ joining ζ to γ gives an extension of γ beyond b_1.

Hence $b_1 = +\infty$.

Similar argument $\Rightarrow a_1 = -\infty$

$\therefore$ $\exp_x(tv)$ defined $\forall\, t \in (-\infty, \infty)$.

Since v is arbitrary, $\exp_x$ defined on whole T_xM.

(2) $\Rightarrow$ (3) trivial

(4) $\Rightarrow$ (1) is standard for metric space.

To prove (3) $\Rightarrow$ (4), we claim

(5) Assume $x \in M$ as in (3), then $\forall\, y \in M$, $\exists$ a minimizing geodesic joining x to y.

Pf of claim (5)

Let $\overline{\mathcal{B}}(r) = \{ y \in M : d(x,y) \leq r \}$

$$\Sigma(r) = \left\{ y \in \overline{\mathcal{B}}(r) : \begin{array}{l} y \text{ is joined to } x \text{ by} \\ \text{a min. geodsic.} \end{array} \right\}$$

Then we need to show

$$\overline{\mathcal{B}}(r) = \Sigma(r), \quad \forall\, r \in [0, \infty)$$

Let $J = \{ r \in [0, \infty) : \overline{\mathcal{B}}(r) = \Sigma(r) \}$

Then we have already shown that

if $r < \delta$ where $\delta > 0$ is given by the "Thm" in the previous section,

then $r \in J$

$\Longrightarrow J \neq \phi.$

Next: since $\exp_x$ defined on whole $T_xM \cong \mathbb{R}^n$

continuous dependence of $\exp_x(tv)$ on v

$\Rightarrow$ J is closed.

To show J is open, we need the following fact

(Ex, or see do Carmo)

(*) $\left[\begin{array}{l} \forall \text{ cpt } K \subset M, \ \exists\, \varepsilon > 0 \text{ s.t.} \\ \forall\, y, z \in K \text{ with } d(y,z) \le \varepsilon, \\ \text{then } \exists \text{ a minimizing geodesic joining } y \,\&\, z. \end{array}\right.$

Note: This is a stronger result than the last Thm in §4.1

(in which one of the points has to be the center.)

Pf of openness: Define $K = \overline{\mathcal{B}}(r)$, $\forall r$

Then $\overline{\mathcal{B}}(r) \subset \exp_x(\overline{B}(r))$

$\Rightarrow \overline{\mathcal{B}}(r)$ cpt. (since $\overline{B}(r)$ cpt in T_xM & $\exp_x$ diffeo.

Applying $(*)$, $\exists\, \varepsilon > 0$ with property stated in $(*)$.

Let $\varepsilon' \in (0, \varepsilon)$ and $y \in \overline{\mathcal{B}}(r+\varepsilon')$.

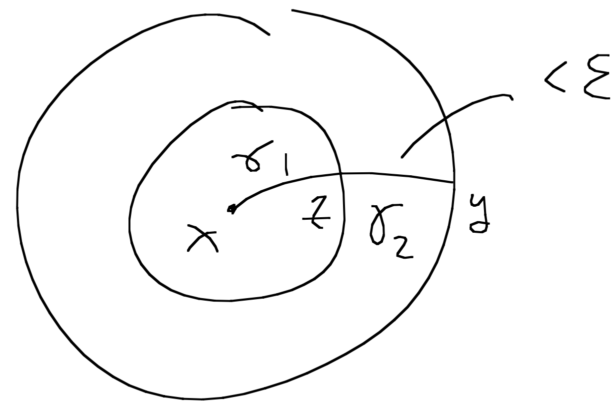
If $y \in \overline{\mathcal{B}}(r)$, then $y \in \Sigma(r) \subset \Sigma(r+\varepsilon')$

$(\because r \in \mathcal{S})$

If $y \in \overline{\mathcal{B}}(r+\varepsilon') \setminus \overline{\mathcal{B}}(r)$, $\exists\, z \in \partial \overline{\mathcal{B}}(r)$

s.t. $d(x,y) = d(x,z) + d(z,y)$

(by using cptness of $\partial \bar{B}(r)$ & definition of $d(x,y)$)



Then $r \in J \Rightarrow$

$\exists$ minimizing geodesic γ_1 joining x & z

On the other hand,

$$d(z,y) = d(x,y) - d(x,z) \leq r + \varepsilon' - r = \varepsilon' < \varepsilon$$

$\overset{(*)}{\Longrightarrow}$ $\exists$ minimizing geodesic γ_2 joining z & y.

Then connecting γ_1 & γ_2, we have a (piecewise smooth) curve joining x & y with

$$\text{length} = d(x,z) + d(z,y) = d(x,y)$$

$\Rightarrow$ it must be a minimizing geodesic.

Therefore $\mathcal{J}$ is open.

Altogether, $\mathcal{J}$ is open, closed, nonempty subset of $[0,\infty) \Rightarrow \mathcal{J} = [0,\infty) \Rightarrow$ claim (5) $\divideontimes$

Pf of (3) $\Rightarrow$ (4)

By claim (5), $\forall$ bounded & closed set K, $\exists A > 0$ s.t. $d(x,k) \le A$, $\forall k \in K$

$\Rightarrow \quad K \subset \exp_x(\bar{B}(A))$

$\Rightarrow \quad K$ is cpt (since $\bar{B}(A)$ is cpt.) $\#$

This completes the proof of Hopf-Rinow Thm.

Pf of Cor 1: Hopf-Rinow $\Rightarrow$ (2) is true

($\Rightarrow$ (3) is true)

$\Rightarrow$ claim (5) is true $\forall x \in M$

$\Rightarrow$ Cor 1 is true $\#$

Ch5 Isometry, Space forms

(M, g) = complete Riemannian manifold (connected)

Def: (M,g) with constant sectional curvature is called a <u>space form</u>.

Thm1: $\forall\, c \in \mathbb{R}$ & $n \geq 2$, $\exists$ unique (up to isometry) <u>simply-connected</u> space form of dimension n and with constant sectional curvature c.

egs (proof later)

- $C = 0$ $(\mathbb{R}^n$, standard flat metric$)$
- $C = +1$ $(S^n$, standard metric$)$
- $C = -1$ $\left(\mathbb{B}^n, \dfrac{4}{\left[1 - \sum_{i=1}^n (x^i)^2\right]^2}(dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n)\right)$

where $\mathbb{B}^n = \{(x^1, \cdots, x^n) : \sum_{i=1}^n (x^i)^2 < 1\}$

(Hyperbolic n-space : unit ball model)

Def: Let M be a submanifold of $\overline{M}$ equipped with the induced metric. Then M is called a <u>totally geodesic submanifold</u> of $\overline{M}$ if a geodesic γ (of $\overline{M}$) tangents to M implies $\gamma \subset M$.

<u>Note</u>: Such a geodesic γ of $\overline{M}$ must be a geodesic of the submanifold M.

<u>egs</u>:

- $\mathbb{R}^k \hookrightarrow \mathbb{R}^n : (x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, 0, \dots, 0)$ gives a totally geodesic submanifold of $\mathbb{R}^n$.
- $S^n \subset \mathbb{R}^{n+1}$ is not a totally geodesic submanifold (since tangent lines to S^n don't stay in S^n.)

Let
- $M \subset \bar{M}$ be a submanifold
- M equipped with induced metric
- $D, \bar{D}$ = Levi-Civita connections of $M, \bar{M}$ respectively

(note: $D_X Y = (\bar{D}_X Y)^{\text{tangential part}}$)

Consider $S(X,Y) = D_X Y - \bar{D}_X Y$, $\forall X, Y \in \Gamma(TM)$

(note: S defined for vector fields on M, not $\bar{M}$)

Facts:
- $S(X_1 + X_2, Y) = S(X_1, Y) + S(X_2, Y)$
- $S(X,Y) = S(Y,X)$
- $\forall f \in C^\infty(M)$, $S(fX, Y) = S(X, fY) = f S(X,Y)$

The last one $\Rightarrow$ S defines a tensor field on M.

<u>Pf of Symmetry</u>: $S(X,Y) - S(Y,X)$

$$= (D_X Y - \bar{D}_X Y) - (D_Y X - \bar{D}_Y X)$$

$$= (D_X Y - D_Y X) - (\bar{D}_X Y - \bar{D}_Y X)$$

$$= [X,Y] - [X,Y] = 0 \quad (D, \bar{D} = \text{Levi-Civita})$$

others are easy (Ex.).

Therefore, we can define a symmetric bilinear form on T_xM, $\forall x \in M$:

$$\forall v, w \in T_xM, \quad S(v,w) = S(V,W) \quad \left(S_x(v,w) = S(V,W)(x) \right)$$

where V, W = extension of v, w.

<u>Def</u>: This S is called the <u>2nd fundamental form</u> of M in $\overline{M}$.

<u>Lemma 2</u>: $M \subset \overline{M}$ totally geodesic

$\iff S \equiv 0$, where $S = 2^{nd}$ f.f. of M in $\overline{M}$

(i.e. $D_X Y = \overline{D}_X Y$)

<u>Pf</u>: ($\Rightarrow$) Let $x \in M$ & $v \in T_x M \subset T_x \overline{M}$

Let γ = geodesic on $\overline{M}$ with

$\gamma(0) = x$, $\gamma'(0) = v$.

$$\Rightarrow \quad \bar{D}_{\gamma'}\gamma' = 0$$

By assumption, γ is also a geodesic of M

$$\Rightarrow \quad D_{\gamma'}\gamma' = 0$$

Therefore $\quad S(v,v) = S(\gamma'(0), \gamma'(0))$

$$= D_{\gamma'}\gamma' - \bar{D}_{\gamma'}\gamma' = 0$$

Symmetry of $S \Rightarrow \quad S(v,w) = 0 \quad \forall\, v, w \in T_x M$.

($\Leftarrow$) Suppose $S \equiv 0$.

Let γ be a geodesic of $\bar{M}$ such that

$$\gamma(0) = x \text{ and } \gamma'(0) = v \in T_x M \subset T_x \bar{M}.$$

By Existence & Uniqueness of geodesic in M,

$\exists\ \xi =$ geodesic of M s.t.

$$\xi(0) = x\ ,\ \xi'(0) = v \in T_xM.$$

Then $S \equiv 0$

$$\Rightarrow\quad \bar{D}_{\xi'(t)}\xi'(t) = D_{\xi'(t)}\xi'(t) = 0 \quad (\xi = \text{geo. of } M)$$

$\Rightarrow$ ξ is also a geodesic of $\bar{M}$

Then uniqueness $\Rightarrow$ $\gamma = \xi \subset M$ ##

<u>Lemma 3</u> Let $M \subset \bar{M}$ be totally geodesic,

$K, \bar{K}$ = sectional curvatures of $M, \bar{M}$ respectively.

Then $\forall\, x \in M$, $\forall$ 2-plane $\Pi \subset T_xM \subset T_x\bar{M}$,

$$K(\Pi) = \bar{K}(\Pi)$$

(Pf: Immediately from Lemma 2)

<u>eg</u>: Let $\gamma : (a,b) \to \bar{M}$ be a smooth curve parametrized by arc-length. Suppose $\exists$ isometry $\varphi : \bar{M} \to \bar{M}$ s.t.

$$\gamma((a,b)) = \{ y \in \bar{M} : \varphi(y) = y \}$$

Then γ is a <u>normalized geodesic</u>.

Pf: We first note that $\forall$ geodesic ξ in $\bar{M}$, $\varphi \circ \xi$ is also a geodesic in $\bar{M}$ (since φ = isom.)

Now $\forall t_0 \in (a,b)$, take a geodesic

$\zeta \subset \bar{M}$ s.t. $\begin{cases} \zeta(0) = \gamma(t_0) \\ \zeta'(0) = \gamma'(t_0) \end{cases}$.

Since $\gamma((a,b)) =$ fixed point set of φ

$d\varphi(\gamma'(t_0)) = \gamma'(t_0)$ (by diff. $\varphi \circ \gamma = \gamma$)

$\Rightarrow d\varphi(\zeta'(0)) = \zeta'(0)$

$\Rightarrow (\varphi \circ \zeta)'(0) = \zeta'(0)$ (since $\varphi(\zeta(0)) = \zeta(0)$)

Uniqueness of geodesic $\Rightarrow \varphi \circ \zeta = \zeta$

$\Rightarrow \zeta \subset \{ y \in \bar{M} : \varphi(y) = y \} = \gamma((a,b))$

$\Rightarrow \gamma$ is normalised geodesic ※

Lemma 4 : The set of fixed points of an isometry is a totally geodesic submanifold. (not necessary connected.)

Pf : Let $\varphi : \bar{M} \to \bar{M}$ be an isometry &

$M = \{ y \in \bar{M} : \varphi(y) = y \}$ be the set of fixed points of φ.

If M is submanifold of $\bar{M}$, then the same argument as in the previous example implies M is totally geodesic. So we only need to show the following claim :

<u>Claim</u>: Let $x \in M$, $B(\delta) = \{ v \in T_x\overline{M} : |v| < \delta \}$

$$B_\delta = \{ y \in \overline{M} : d(x,y) < \delta \}$$

where $\delta > 0$ small enough s.t.

$$\exp_x : B(\delta) \longrightarrow B_\delta \text{ is a diffeomorphism}$$

$$\left(\Rightarrow \quad B_\delta = \exp_x B(\delta) \right)$$

Let $\mathcal{F} \subset T_x\overline{M}$ be a linear subspace defined by

$$\mathcal{F} = \{ v \in T_x\overline{M} : d\varphi(v) = v \}$$

Then

$$M \cap B_\delta = \exp_x(\mathcal{F} \cap B(\delta)).$$

Hence M is submanifold of $\overline{M}$.

Pf of Claim :

(1) $M \cap B_\delta \subset \exp_x(\mathcal{F} \cap B(\delta))$

Pf : Let $y \in M \cap B_\delta \subset B_\delta$

$\Rightarrow \exists\, v \in B(\delta)$ s.t. $\exp_x v = y$.

Let $\gamma(t) = \exp_x(tv) : [0,1] \to \overline{M}$

be the unique minimizing geodesic joining

x to y.

Since $x, y \in M$, we have $\varphi(x) = x$ & $\varphi(y) = y$

$\Rightarrow \varphi \circ \gamma$ is also a minimizing geodesic joining

x to y.

Uniqueness

$$\Longrightarrow \quad \varphi \circ \gamma = \gamma$$

$$\Longrightarrow \quad d\varphi(v) = v$$

$$\Longrightarrow \quad v \in \mathcal{J}$$

$$\therefore \quad y = \exp_x v \in \exp_x(\mathcal{J} \cap B(\delta))$$

This proves (1).

(2) $\exp_x(\mathcal{J} \cap B(\delta)) \subset M \cap B_\delta$

Pf: Let $y \in \exp_x(\mathcal{J} \cap B(\delta))$.

Then $\exists\ v \in \mathcal{J} \cap B(\delta)$ such that

$$y = \exp_x v.$$

Let $\gamma(t) = \exp_x(tv) : [0,1] \to \overline{M}$ be the unique minimizing geodesic joining x to y.

Since $v \in \mathcal{F}$, $d\varphi(\gamma'(0)) = \gamma'(0)$

$\Rightarrow$ $\varphi \circ \gamma$ & γ have the same initial values.

$\xrightarrow{\text{Uniqueness}}$ $\varphi \circ \gamma = \gamma$

$\Rightarrow$ $y = \gamma(1) = \varphi(\gamma(1)) = \varphi(y)$

$\Rightarrow$ $y \in M \cap B_\delta$. ※

<u>Lemma 5</u>: $S^n \subset \mathbb{R}^{n+1}$ has constant sectional curvature $+1$, $\forall n \geq 2$.

<u>Pf</u>: "$n=2$" is proved in undergrad DG (ex)

If $n \geq 3$, define

$$\tilde{\varphi} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$

$$(x^1, x^2, x^3, x^4 \dots x^{n+1}) \mapsto (x^1, x^2, x^3, -x^4, \dots, -x^{n+1})$$

Then $|\tilde{\varphi}(x)| = |x|$ (Euclidean norm)

Hence $\tilde{\varphi}$ induces an isometry

$$\varphi : S^n \to S^n.$$

The fixed points set

$$M = \{x \in S^n : \varphi(x) = x\}$$

$$= \{(x^1, x^2, x^3, 0, \dots, 0) : (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}$$

$= S^2$

is a totally geodesic submanifold.

Hence $K_{S^n}(\Pi) = K_{S^2}(\Pi) = +1$,

$\forall$ 2-plane $\Pi \subset T_x S^2$, where $x \in (x^1, x^2, x^3, 0, \cdots, 0)$.

Repeat the argument for any 3 indices $i, j, k \in \{1, \cdots, n+1\}$

and the fact S^n is invariant under rotation,

we have proved that $K_{S^n} \equiv +1$. #