MATH5011 Real Analysis I

Exercise 3 Suggested Solution

Standard notations are in force.

(1) Prove the conclusion of Lebsegue's dominated convergence theorem still holds when the condition "$\{f_k\}$ converges to f a.e." is replaced by the condition " $\{f_k\}$ converges to f in measure".

Solution: Suppose on the contrary that $\int |f_k - f| d\mu$ does not tend to zero. By considering the limit supremum of the sequence, we can find a positive constant M and a subsequence $\{f_{n_j}\}$ such that

$$\int |f_{n_j} - f| d\mu \geq M$$

for all j. By Prop 1.17, a subsequence $\{g_k\}$ of $\{f_{n_j}\}$ converges to f a.e.. By Lebsegue's dominated convergence theorem, we have

$$0 = \lim_{k\to\infty} \int |g_k - f| d\mu \geq M > 0,$$

contradiction holds.

(2) Let $f_n, n \geq 1$, and f be real-valued measurable functions in a finite measure space. Show that $\{f_n\}$ converges to f in measure if and only if each subsequence of $\{f_n\}$ has a subsubsequence that converges to f a.e..

Solution: Let μ be the measure in a finite measure space. If $\{f_n\}$ converges to f in measure, then every subsequence $\{f_{n_k}\}$ also converges to f in measure. By Prop 1.17, the subsequence $\{f_{n_k}\}$ has a sub-subsequence converging to f a.e..

Now suppose that each subsequence of $\{f_n\}$ has a sub-subsequence that converges to f a.e.. Assume that f_n does not converge to f in measure. By considering the limit supremum, there are positive ρ , M and subsequence $\{f_{n_j}\}$ such that, $\mu(\{x : |f_{n_j}(x) - f(x)| \geq \rho\}) \geq M$, for all j. However, a subsequence $\{g_k\}$ of $\{f_{n_j}\}$ converges to f a.e. and by Prop 1.18, $\{g_k\}$ converges to f in measure and

$$0 = \lim_{k\to\infty} \mu(\{x : |g_k(x) - f(x)| \geq \rho\}) \geq M > 0,$$

which is impossible. Hence $\{f_n\}$ converges to f in measure.

(3) Let X be a metric space and $\mathcal{C}$ be a subset of $\mathcal{P}_X$ containing the empty set and X. Assume that there is a function $\rho : \mathcal{C} \to [0, \infty]$ satisfying $\rho(\phi) = 0$. For each $\delta > 0$, show that (a)

$$\mu_\delta(E) = \inf \Big\{ \sum_k \rho(C_k) : E \subset \bigcup_k C_k, \quad \text{diameter}(C_k) \leq \delta \Big\}$$

is an outer measure on X, and (b) $\mu(E) = \lim_{\delta\to 0} \mu_\delta(E)$ exists and is also an outer measure on X.

Solution:

(a) To see that (i) is satisfied, observe that the empty set ϕ is contained in any $\mathcal{C}$, so that $\mu(\phi) = \rho(\phi) = 0$. Let $A \subset \bigcup_{j=1}^\infty A_j$. Suppose that $\sum_j \mu(A_j) < \infty$ (otherwise there is nothing to prove). For each $\varepsilon > 0$, we can find $C_k^j, k \geq 1$, in $\mathcal{C}$ such that diameter$(C_k^j) \leq \delta, A_j \subset \bigcup_k C_k^j$ and

$\sum_k \varphi(C_k^j) \le \mu(A_j) + \varepsilon/2^j$. As $\{C_k^j\}$ covers A and diameter$(C_k^j) \le \delta$,

$$\begin{aligned}\mu(A) &\le \sum_{j,k} \varphi(C_k^j) \\ &\le \sum_j \sum_k \varphi(C_k^j) \\ &\le \sum_j \Big(\mu(A_j) + \frac{\varepsilon}{2^j}\Big) \\ &\le \sum_j \mu(A_j) + \varepsilon,\end{aligned}$$

and (ii) holds after letting ε tend to 0.

(b) Since $\mu_\delta(E)$ is decreasing in δ, $\mu(E) = \lim_{\delta\to 0} \mu_\delta(E)$ exists. Since $\mu_\delta(\phi) = 0, \mu(\phi) = 0$. Hence (i) is satisfied. Let $A \subset \bigcup_{j=1}^\infty A_j$. Suppose that $\sum_j \mu(A_j) < \infty$. One has

$$\mu(A) \le \mu_\delta(A) \le \sum_j \mu_\delta(A_j).$$

It follows from monotonicity that

$$\mu(A) \le \liminf_{\delta\to 0} \sum_j \mu_\delta(A_j) = \sum_j \mu(A_j).$$

(4) Here we consider an application of Caratheodory's construction. An *algebra* $\mathcal{A}$ on a set X is a subset of $\mathcal{P}_X$ that contains the empty set and is closed under taking complement and finite union. A *premeasure* $\mu : \mathcal{A} \to [0,\infty]$ is a finitely additive function which satisfies: $\mu(\phi) = 0$ and $\mu(\bigcup_{k=1}^\infty E_k) = \sum_{k=1}^\infty \mu(E_k)$ whenever E_k are disjoint and $\bigcup_{k=1}^\infty E_k \in \mathcal{A}$. Show that the premeasure μ can be extended to a measure on the σ-algebra generated by $\mathcal{A}$. Hint: Define the outer measure

$$\overline{\mu}(E) = \inf\Big\{ \sum_k \mu(E_k) : E \subset \bigcup_k E_k, E_k \in \mathcal{A}\Big\}.$$

This is called Hahn-Kolmogorov theorem.

Solution: We follow the proof from Terence Tao's book "Introduction To Measure Theory". Define $\overline{\mu}$ as above and obviously $\overline{\mu}$ is an outer measure on power set of X. By Caratheodory's construction, we get a measure defined on a σ- algebra M. We claim that $\mathcal{A} \subseteq M$, let $E \in \mathcal{A}$ and $C \subseteq X$ such that $\overline{\mu}(C) < \infty$, for all $\varepsilon > 0$, there is $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ covering C such that

$$\overline{\mu}(C) + \varepsilon > \sum_{i=1}^{\infty} \mu(E_i).$$

As $\{E_i \cap E\}_{i=1}^{\infty}$ and $\{E_i \setminus E\}_{i=1}^{\infty}$ are subset of $\mathcal{A}$ and cover $C \cap E$ and $C \setminus E$ respectively, we have,

$$\overline{\mu}(C \cap E) \leq \sum_{i=1}^{\infty} \mu(E_i \cap E),$$

and

$$\overline{\mu}(C \setminus E) \leq \sum_{i=1}^{\infty} \mu(E_i \setminus E).$$

Using the fact that μ is a premeasure, $\mu(E_i \cap E) + \mu(E_i \setminus E) = \mu(E_i)$. Summing over i, we know that E is in M and M contains the σ algebra generated by $\mathcal{A}$. Now we try to show that the measure induced extends μ, obviously by definition $\overline{\mu}(E) \leq \mu(E)$. Let $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ covering E. Without affecting the countable union, we may make $\{E_i\}_{i=1}^{\infty}$ disjoint and obtain $\{B_i\}_{i=1}^{\infty}$. Furthermore, by taking intersection with E, we have

$$\bigcup_{i=1}^{\infty} B_i \cap E = E$$

and

$$\sum_{i=1}^{\infty} \mu(E_i) \geq \sum_{i=1}^{\infty} \mu(B_i) \geq \sum_{i=1}^{\infty} \mu(B_i \cap E) = \mu(E),$$

where the last equality follows from the condition of μ. Hence $\overline{\mu}(E) \geq \mu(E)$

and the measure extends μ.

(5) Let $(X, \overline{\mathcal{M}}, \overline{\mu})$ be the completion of $(X, \mathcal{M}, \mu)$ as described in Ex 1. Show that $\overline{\mathcal{M}}$ is the σ-algebra generated by $\mathcal{M}$ and all subsets of measure zero sets in $\mathcal{M}$.

Solution: Let $\mathcal{M}_1$ be the σ-algebra generated by $\mathcal{M}$ and all subsets of measure zeros sets in $\mathcal{M}$.

By definition, $\overline{\mathcal{M}}$ contains all the sets in $\mathcal{M}$ and all subsets of measure zero sets in $\mathcal{M}$. Since $\mathcal{M}_1$ is the smallest such σ-algebra, we have $\mathcal{M}_1 \subset \overline{\mathcal{M}}$.

To prove that $\overline{\mathcal{M}} \subset \mathcal{N}$, we let $E \in \overline{\mathcal{M}}$. Then there exist $A, B \in \mathcal{M}$ such that $A \subset E \subset B$ and $\mu(B \setminus A) = 0$. Then $E \setminus A \subset B \setminus A$ is a subset of a measure zero set. Now $E = A \cup (E \setminus A)$ is a union of a set in $\mathcal{M}$ and a subset of a measure zero set. Hence, $E \in \mathcal{M}_1$.

(6) Find a complete measure space $(X, \mathcal{M}, \mu)$ in which $\mathcal{M} \subsetneq \mathcal{M}_C$. This problem is related to Theorem 2.2.

Solution: Take X a set containing more than two elements and set

$$m(A) = \begin{cases} 0, & \text{if } A = \phi, \\ \infty, & \text{otherwise.} \end{cases}$$

Take the initial σ-algebra to be $\mathcal{M} = \{\phi, X\}$. Then the σ- algebra due to Caratheodory construction is $\mathcal{M}_C = \mathcal{P}(X)$, the power set of X.

(7) Let X be a metric space and $C(X)$ the collection of all continuous real-valued functions in X. Let $\mathcal{A}$ consist of all sets of the form $f^{-1}(G)$ which $f \in C(X)$ and G is open in $\mathbb{R}$. The "Baire σ-algebra" is the σ-algebra generated by $\mathcal{A}$. Show that the Baire σ-algebra coincides with the Borel σ-algebra $\mathcal{B}$.

Solution: It is clear that the Baire σ-algebra is contained in $\mathcal{B}$, since $f^{-1}(G)$ is an open whenever f is continuous and G is open in $\mathbb{R}$. Conversely, We prove the inclusion using the metric d function on the metric space X. Let U be arbitrary open set in X, F be its complement in X, and consider the function $f(x) \equiv \inf\{d(x,y) : y \in F\}$. As F is closed, $f(x)$=0 $\Leftrightarrow$ x is not in U, obviously f is in $C(X)$ and $U = f^{-1}(R^+)$. Hence every open U is in the Baire σ-algebra.

(8) Show that the open ball $\{(x,y) : x^2 + y^2 < 1\}$ in $\mathbb{R}^2$ cannot be expressed as a disjoint union of open rectangles. Hint: What happens to the boundary of any of these rectangles? This is in contrast with the one-dimensional case.

Solution: Suppose the open ball $B = \{(x,y) : x^2+y^2 < 1\}$ can be expressed as a disjoint union of open rectangles. Pick any R in the union. Then there is a point $x \in B$ on the boundary ∂R of R. Since R is open, $x \in B \setminus R$. This implies $x \in \widetilde{R}$ for some other open rectangle $\widetilde{R}$ in the union. Now $R \cap \widetilde{R} \neq \phi$, showing that the union cannot be disjoint.

(9) Show that every open set in $\mathbb{R}^n$ can be expressed as a countable almost disjoint union of rectangles. Here almost disjoint means the interiors of rectangles are mutually disjoint.

Solution: Let G be an open set in $\mathbb{R}^n$. Let $\mathcal{S}_j, j \geq 1$, be the collection of all closed cubes of sides in the form $[k/2^j, (k+1)/2^k], k \in \mathbb{Z}$, that are contained in G. We select a subcollection $\mathcal{T}_j$ from $\mathcal{S}_j$ in the following manners. First, let $\mathcal{T}_1$ be $\mathcal{S}_1$ and then $\mathcal{T}_2$ be those cubes in $\mathcal{S}_2$ which are disjoint from the cubes in $\mathcal{T}_1$. Next, $\mathcal{T}_3$ are cubes in $\mathcal{S}_3$ which are disjoint from the cubes in $\mathcal{T}_1$ and $\mathcal{T}_2$. Keep doing this we get $\mathcal{T}_j, j \geq j_0$. (There are no closed cubes of side length $1/2^{j_0-1}$ contained in G.) We claim the collection of all cubes in $\mathcal{T}_j$ satisfies our requirement.

First of all, they are almost disjoint and contained in G. Next, let $x \in G$. Since G is open, x would belong to some $\mathcal{S}_j$. So let j_1 be the first j with this property and let $x \in R_1$ where $R_1 \in \mathcal{S}_{j_1}$. Then R_1 is not contained in any

$\mathcal{T}_m, m < j_1$, so R_1 is selected into $\mathcal{T}_{j_1}$. This shows that the union of all cubes in $\mathcal{T}_j, j \geq j_1$, satisfies our requirement.

The following problems are concerned with the Lebesgue measure. Let $R = I_1 \times I_2 \times \cdots \times I_n$, I_j bounded intervals (open, closed or neither), be a rectangle in $\mathbb{R}^n$.

(9) For a rectangle R in $\mathbb{R}^n$, define its "volume" to be

$$|R| = (b_1 - a_1) \times (b_2 - a_2) \times \cdots \times (b_n - a_n)$$

where b_i, a_i are the right and left endpoints of I_j. Show that

(a) if $R = \bigcup_{k=1}^{N} R_k$ where R_k are almost disjoint, then

$$|R| = \sum_{k=1}^{N} |R_k|.$$

(b) If $R \subset \bigcup_{k=1}^{N} R_k$, then

$$|R| \leq \sum_{k=1}^{N} |R_k|.$$

Solution: (a) Take $n = 2$ for simplicity. Each rectangle is of the form $[a, b] \times [c, d]$. We order the endpoints of the x-coordinates of all rectangles $R_1, \cdots, R_N$ into $a_1 < a_2 < \cdots < a_n$ and y-coordinates into $b_1 < b_2 < \cdots < b_m$. This division breaks R into an almost disjoint union of subrectangles $R_{j,k} = [a_j, a_{j+1}] \times [b_k, b_{k+1}]$. Note that each $R_{j,k}$ is contained in exactly one R_l and each R_l is an almost disjoint union of subrectangles from this division.

We have

$$\begin{aligned}
|\mathbb{R}| &= (a_n - a_1)(b_m - b_1) \\
&= (a_n - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_2 - a_1)(b_m - b_{m-1} + b_{m-1} + \cdots + b_2 - b_1) \\
&= \sum_{j,k} |R_{j,k}| \\
&= \sum_{l} \sum_{R_{j,k} \subseteq R_l} |R_{j,k}| \\
&= \sum_{l} |R_l|,
\end{aligned}$$

(b) The proof is similar to that of (a), but now we need to subdivide R_j and R together. Now, $R \subseteq \bigcup_{j=1}^{N} R_j$. We order all x-coordinates of R_j, R into $a_1 < a_2 < \cdots < a_N$ and and y-coordinates into $b_1 < b_2 < \cdots < b_M$. Then R is the union of parts of $R_{k,j}$

$$\begin{aligned}
|R| &= \sum_{R_{j,k} \subseteq R} |R_{j,k}| \\
&\leq \sum_{j,k} |R_{j,k}| \\
&\leq \sum_{j} |R_j|,
\end{aligned}$$

where the last inequality follows from the fact that each $R_{k,j}$ is contained in some R_j.

(10) Let $\mathcal{R}$ be the collection of all closed cubes in $\mathbb{R}^n$. A closed cube is of the form $I \times \cdots \times I$ where I is a closed, bounded interval.

(a) Show that $(\mathcal{R}, |\cdot|)$ forms a gauge, and thus it determines a complete measure $\mathcal{L}^n$ on $\mathbb{R}^n$ called the *Lebesgue measure*.

(b) $\mathcal{L}^n(R) = |R|$ where R is a cube, closed or open.

(c) For any set E and $x \in \mathbb{R}^n$, $\mathcal{L}^n(E+x) = \mathcal{L}^n(E)$. Thus the Lebsegue measure is translational invariant.

(d) Show that the Lebesgue measure is a Borel measure.

Hint: Use Caratheodory's criterion.

(e) Show that for every $E \subset \mathbb{R}^n$,

$$\mathcal{L}^n(E) = \inf \big\{ \mathcal{L}^n(G) : \ E \subset G, \ G \text{ open} \big\}.$$

It means that the Lebsegue measure is outer regular.

Solution:

(a) We clearly have

$$\inf_{R \in \mathcal{R}} |R| = 0 \quad \text{and} \quad \bigcup_{R \in \mathcal{R}} R = \mathbb{R}^n.$$

Hence, $\mathcal{R}$ forms a gauge.

Define μ by

$$\mu(A) = \inf \left\{ \sum_{j=1}^{\infty} |G_j| : G_j \in \mathcal{R} \right\}.$$

We check that μ is an outer measure.

- Clearly, $\mu(\emptyset) = 0$.
- Suppose $\{E_j : j \in \mathcal{N}\}$ are given and write $E = \bigcup_{j \in \mathbb{N}} E_j$. Let $\varepsilon > 0$. Choose $G_{jk} \in \mathcal{R}$ such that

$$\mu(E_j) + 2^{-j}\varepsilon > \sum_{k \in \mathbb{N}} |G_{jk}|.$$

Then $\{G_{jk} : j, k \in \mathbb{N}\}$ is a countable cover for E. We have

$$\mu(E) \le \sum_{j,k\in\mathbb{N}} |G_{jk}| < \sum_{j\in\mathbb{N}} \left(\mu(E_j) + 2^{-j}\varepsilon\right) = \sum_{j\in\mathbb{N}} \mu(E_j) + \varepsilon.$$

Taking $\varepsilon \to 0$, we have

$$\mu(E) \le \sum_{j\in\mathbb{N}} \mu(E_j).$$

Following the Carathéodory's construction, we obtain a complete measure.

(b) By (9)(b), it suffices to show that $R \subseteq \bigcup^{\infty} R_j \Rightarrow |R| \le \sum^{\infty} |R_j|$. We replace $R_j = [a_j, b_j] \times [c_j, d_j]$ by $\acute{R_j} = (a_j - \frac{\varepsilon}{2^j}, b_j + \frac{\varepsilon}{2^j}) \times (c_j - \frac{\varepsilon}{2}, d_j + \frac{\varepsilon}{2})$. Since $\{\acute{R_j}\}$ is an open cover of R and R is compact, there exists a finite subcover $\acute{R}_{j_1}, \cdots, \acute{R}_{j_M}$. By (9)(b)

$$|R| \le \sum_{k=1}^{M} |\acute{R}_{j_k}| \le \sum_{j,k} |R_{j,k}| \le \sum_{j} |R_j| + C\varepsilon,$$

C depends on n only. Let $\varepsilon \to 0$,

$$|R| \le \sum_{j} |R_j|,$$

which shows that

$$|R| = \inf\{\sum_{1}^{\infty} |R_j| : R \subseteq \bigcup_{j=1}^{\infty} R_j, R_j \text{ closed cube}\}.$$

Therefore,

$$\mathcal{L}^n(R) = |R|$$

where R is a closed cube.

In order to show it also holds for any cube, it suffices to show $\mathcal{L}^n(F) = 0$ whenever F is a face of R. First, $\mathcal{L}^n(F) \leq \mathcal{L}^n(R) < \infty$. Let N be any number> 1 and $x_1 = (a_1, 0), \cdots, x_N = (a_N, 0)$ be distinct point. Consider $F + x_j$. For small a_j, $F + x_j$ can be chosen to sit inside R, then $\bigcup(F + x_j) \subset R$. By (c), $N\mathcal{L}^n(F) = \sum \mathcal{L}^n(F + x_j) = L^n(\bigcup(F + x_j)) \leq \mathcal{L}^n(R) \Rightarrow \mathcal{L}^n(F) \leq \dfrac{\mathcal{L}^n(R)}{N} \to 0$ as $N \to \infty$.

(c) Result follows directly from definition.

(d) Use Caratheodory criterion,

$\vdash \mathcal{L}^n(A \cup B) = \mathcal{L}^n(A) + \mathcal{L}^n(B)$ if $dist(A, B) = \rho > 0$

Pf: Let $\{R_j\}$ be $A \cup B \subseteq \bigcup R_j$,

$$\mathcal{L}^n(A \cup B) + \varepsilon \geq \sum |R_j|.$$

We subdivide every R_j into small cubes $\tilde{R_k}$ of diameter δ,

$$\sum_{j=1}^{N} |R_j| = \sum_{k=1}^{M} |\tilde{R_k}|.$$

Divide $\tilde{R_k}$ into 3 classes: First $\mathcal{R}_1$ those intersect A, $\mathcal{R}_2$ those intersect B and $\mathcal{R}_3$ intersect neither.

For $\delta \leq \frac{\rho}{2}, \mathcal{R}_1 \cap \mathcal{R}_2 = \phi$. Then

$$\mathcal{L}^n(A) \leq \sum_{\mathcal{R}_1} |\tilde{R_k}|,$$

$$\mathcal{L}^n(B) \leq \sum_{\mathcal{R}_2} |\tilde{R_k}|.$$

Therefore

$$\mathcal{L}^n(A \cup B) + \varepsilon \geq \mathcal{L}^n(A) + \mathcal{L}^n(B).$$

Let $\varepsilon \to 0$, done

(e) If $\mathcal{L}^n(E) = \infty$, $E \subseteq \mathbb{R}^n$, $\mathcal{L}^n(\mathbb{R}^n) = \infty$. If $\mathcal{L}^n(E) < \infty, \forall \varepsilon > 0, \exists \{R_j\}, E \subseteq \bigcup R_j$ s.t.

$$\mathcal{L}^n(E) + \varepsilon \geq \sum |R_j|.$$

$R_j = [a_j, b_j] \times [c_j, d_j], \tilde{R}_j = (a_j - \frac{\rho}{2^j}, b_j + \frac{\rho}{2^j}) \times (c_j - \frac{\rho}{2^j}, d_j + \frac{\rho}{2^j})$. $G = \bigcup \tilde{R}_j$ is open and contains E and

$$\mathcal{L}^n(G) \leq \sum \mathcal{L}^n(\tilde{R}_j) \leq \sum |R_j| + C\rho$$

C is a dimensional constant. Therefore

$$\mathcal{L}^n(E) + \varepsilon \geq \sum |R_j| \geq \mathcal{L}^n(G) - C\rho.$$

Take $\rho = \varepsilon$,

$$\mathcal{L}^n(E) + (1 + C)\varepsilon \geq \mathcal{L}^n(G),$$

Which show that

$$\mathcal{L}^n(E) = \inf\{\mathcal{L}^n(G) : E \subseteq G \text{ open}\}.$$