MATH5011 Exercise 3

Standard notations are in force.

(1) Prove the conclusion of Lebsegue's dominated convergence theorem still holds when the condition "$\{f_k\}$ converges to f a.e." is replaced by the condition " $\{f_k\}$ converges to f in measure".

(2) Let $f_n, n \geq 1$, and f be real-valued measurable functions in a finite measure space. Show that $\{f_n\}$ converges to f in measure if and only if each subsequence of $\{f_n\}$ has a subsubsequence that converges to f a.e..

(3) Let X be a metric space and $\mathcal{C}$ be a subset of $\mathcal{P}_X$ containing the empty set and X. Assume that there is a function $\rho : \mathcal{C} \to [0, \infty]$ satisfying $\rho(\phi) = 0$. For each $\delta > 0$, show that (a)

$$\mu_\delta(E) = \inf \Big\{ \sum_k \rho(C_k) : E \subset \bigcup_k C_k, \quad \text{diameter}(C_k) \leq \delta \Big\}$$

is an outer measure on X, and (b) $\mu(E) = \lim_{\delta \to 0} \mu_\delta(E)$ exists and is also an outer measure on X.

(4) Here we consider an application of Caratheodory's construction. An *algebra* $\mathcal{A}$ on a set X is a subset of $\mathcal{P}_X$ that contains the empty set and is closed under taking complement and finite union. A *premeasure* $\mu : \mathcal{A} \to [0, \infty]$ is a finitely additive function which satisfies: $\mu(\phi) = 0$ and $\mu(\bigcup_{k=1}^\infty E_k) = \sum_{k=1}^\infty \mu(E_k)$ whenever E_k are disjoint and $\bigcup_{k=1}^\infty E_k \in \mathcal{A}$. Show that the premeasure μ can be extended to a measure on the σ-algebra generated by $\mathcal{A}$. Hint: Define the outer measure

$$\overline{\mu}(E) = \inf \Big\{ \sum_k \mu(E_k) : E \subset \bigcup_k E_k, E_k \in \mathcal{A} \Big\}.$$

This is called Hahn-Kolmogorov theorem.

(5) Let $(X, \overline{\mathcal{M}}, \overline{\mu})$ be the completion of $(X, \mathcal{M}, \mu)$ as described in Ex 1. Show that $\overline{\mathcal{M}}$ is the σ-algebra generated by $\mathcal{M}$ and all subsets of measure zero sets in $\mathcal{M}$.

(6) Find a complete measure space $(X, \mathcal{M}, \mu)$ in which $\mathcal{M} \subsetneq \mathcal{M}_C$. This problem is related to Theorem 2.2.

(7) Let X be a metric space and $C(X)$ the collection of all continuous real-valued functions in X. Let $\mathcal{A}$ consist of all sets of the form $f^{-1}(G)$ which $f \in C(X)$ and G is open in $\mathbb{R}$. The "Baire σ-algebra" is the σ-algebra generated by $\mathcal{A}$. Show that the Baire σ-algebra coincides with the Borel σ-algebra $\mathcal{B}$.

(8) Show that the open ball $\{(x, y) : x^2 + y^2 < 1\}$ in $\mathbb{R}^2$ cannot be expressed as a disjoint union of open rectangles.
Hint: What happens to the boundary of any of these rectangles? This is in contrast with the one-dimensional case.

(9) Show that every open set in $\mathbb{R}^n$ can be expressed as a countable almost disjoint union of rectangles. Here almost disjoint means the interiors of rectangles are mutually disjoint.

The following problems are concerned with the Lebesgue measure. Let $R = I_1 \times I_2 \times \cdots \times I_n$, I_j bounded intervals (open, closed or neither), be a rectangle in $\mathbb{R}^n$.

(9) For a rectangle R in $\mathbb{R}^n$, define its "volume" to be

$$|R| = (b_1 - a_1) \times (b_2 - a_2) \times \cdots \times (b_n - a_n)$$

where b_i, a_i are the right and left endpoints of I_j. Show that

(a) if $R = \bigcup_{k=1}^{N} R_k$ where R_k are almost disjoint (that's, their interiors are pairwise disjoint), then

$$|R| = \sum_{k=1}^{N} |R_k|.$$

(b) If $R \subset \bigcup_{k=1}^{N} R_k$, then

$$|R| \leq \sum_{k=1}^{N} |R_k|.$$

(10) Let $\mathcal{R}$ be the collection of all closed cubes in $\mathbb{R}^n$. A closed cube is of the form $I \times \cdots \times I$ where I is a closed, bounded interval.

(a) Show that $(\mathcal{R}, |\cdot|)$ forms a gauge, and thus it determines a complete measure $\mathcal{L}^n$ on $\mathbb{R}^n$ called the *Lebesgue measure.*

(b) $\mathcal{L}^n(R) = |R|$ where R is a cube, closed or open.

(c) For any set E and $x \in \mathbb{R}^n$, $\mathcal{L}^n(E + x) = \mathcal{L}^n(E)$. Thus the Lebsegue measure is translational invariant.

(d) Show that the Lebesgue measure is a Borel measure.

Hint: Use Caratheodory's criterion.

(e) Show that for every $E \subset \mathbb{R}^n$,

$$\mathcal{L}^n(E) = \inf\left\{\mathcal{L}^n(G) : \ E \subset G, \ G \text{ open}\right\}.$$

It means that the Lebsegue measure is outer regular.