# Suggested Solutions to Test

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**1.** Let $c_0$ be the null sequence space, that is $c_0 := \{(x_n)_{n=1}^\infty : x_n \in \mathbb{R}; \lim_{n\to+\infty} x_n = 0\}$ and is endowed with sup-norm. Define an operator $T : c_0 \to c_0$ by

$$T(x) = (x_2, x_3, \cdots)$$

for $x = (x_1, x_2, x_3, \cdots) \in c_0$.

(a) State the definition of adjoint operator of a bounded linear operator between two normed spaces.

(b) Show that $T$ is bounded and find its norm.

(c) Recall that $c_0^* = \ell^1$. Find the adjoint operator $T^* : \ell^1 \to \ell^1$ of $T$.

**Proof.**

(a) Let $T$ be a bounded linear operator between two normed space $X$ and $Y$. Then the adjoint operator $T^* : Y^* \to X^*$ of $T$ is defined by

$$(T^* f)(x) = f(Tx), \forall f \in X^*, x \in X,$$

where $X^*$ and $Y^*$ are the dual spaces of $X$ and $Y$, respectively.

(b) Let $x = (x_n)_{n=1}^\infty \in c_0$. Then, it follows from the definition of $T$ that

$$\|T(x)\|_\infty = \sup_{n\geq 2} |x_n| \leq \sup_{n\geq 1} |x_n| = \|x\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the sup-norm in $c_0$. So, $T$ is bounded and $\|T\| \leq 1$.
On the other hand, we choose $\tilde{x} = (0, 1, 0, 0, \cdots)$ with only $x_2 = 1$, others 0. Then $\|\tilde{x}\|_\infty = 1$ and $T\tilde{x} = (1, 0, 0, \cdots)$ which yields that $\|T\tilde{x}\|_\infty = 1$. So, $\|T\| := \sup_{\|x\|_\infty \neq 0} \frac{\|Tx\|_\infty}{\|x\|_\infty} \geq \frac{\|T\tilde{x}\|_\infty}{\|\tilde{x}\|_\infty} = 1$.
Therefore, $T$ is bounded with norm to be 1.

(c) For any $\xi = (\xi_n)_{n=1}^\infty \in c_0^* = \ell^1$ and $x = (x_n)_{n=1}^\infty$, it follows from $(a)$ that

$$(T^*\xi)(x) = \xi(Tx) = \sum_{n=1}^\infty \xi_n x_{n+1}, \quad \text{since} \quad Tx = (x_2, x_3, \cdots).$$

Set $\eta = (\eta_n)_{n=1}^\infty := (0, \xi_1, \xi_2, \cdots)$, it is obvious that $\eta \in \ell^1$, since $\xi \in \ell^1$. Then

$$(T^*\xi)(x) = \sum_{n=1}^\infty \eta_n x_n = \eta(x).$$

By the arbitrary of $x$, we have proved that, for any $\xi = (\xi_n)_{n=1}^\infty \in c_0^* = \ell^1$,

$$T^*\xi = \eta,$$

where $\eta = (0, \xi_1, \xi_2, \cdots)$. □

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**2.** Let $X$ and $Y$ be Banach spaces. Let $T : X \to Y$ be a linear operator. For each element $x \in X$, define a norm $\|\cdot\|_T$ on $X$ by

$$\|x\|_T := \|x\| + \|Tx\|$$

for $x \in X$.

(a) Show that the norm $\|\cdot\|_T$ is equivalent to the original norm on $X$ if and only if $T$ is bounded.

(b) State the definition of a closed operator.

(c) Show that if $T$ is a closed operator defined as above, then $\|\cdot\|_T$ is also a Banach norm on $X$.

**Proof.**

(a) Note that it is easy to check that $\|\cdot\|_T$ is indeed a norm on $X$.
Assume that $\|\cdot\|_T$ is equivalent to the original one $\|\cdot\|$, i.e. there exist two positive constants $a, b > 0$ such that $a\|x\| \leq \|x\|_T \leq b\|x\|, \forall x \in X$. Then,

$$\|x\|_T := \|x\| + \|Tx\| \leq b\|x\| \leq (b+1)\|x\|,$$

which yields that $\|Tx\| \leq b\|x\|, \forall x \in X$. So, $T$ is bounded.
On the other hand, assume $T$ is bounded, i.e. $\|Tx\| \leq \|T\|\|x\|$. It follows that

$$\|x\| \leq (\|x\|_T :=)\|x\| + \|Tx\| \leq (1 + \|T\|)\|x\|.$$

So, $\|\cdot\|_T$ is equivalent to $\|\cdot\|$.

(b) Let $T$ be a bounded linear operator between two normed spaces $X$ and $Y$. Then $T$ is a closed operator if its graph $G(T) := \{(x,y)|x \in X, y = Tx\}$ is closed in the normed space $X \times Y$ endowed with norm $\|(x,y)\| = \|x\| + \|y\|, \forall x \in X, y \in Y$.

(c) Let $T$ be a closed operator. Let $\{x_n\}_{n=1}^{\infty}$ be any Cauchy sequence in the normed space $(X, \|\cdot\|_T)$, i.e.

$$\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}^+, \quad s.t. \quad \|x_n - x_m\|_T = \|x_n - x_m\| + \|Tx_n - Tx_m\| < \varepsilon, \forall n, m > N(\varepsilon). \quad (\star)$$

Then, $\lim\limits_{n \to +\infty} \|x_n - x\|$, for some $x \in X$, and $\lim\limits_{n \to +\infty} \|Tx_n - y\| = 0$, for some $y \in Y$, since $\{x_n\}$ and $\{Tx_n\}_{n=1}^{\infty}$ are Cauchy sequences of Banach space $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ respectively. Since $T$ is a closed operator, $(x_n, Tx_n) \to (x, y) \in G(T)$ as $n \to +\infty$,i.e. $y = Tx$. Let $m \to +\infty$ in $(\star)$, we have

$$\|x_n - x\|_T = \|x_n - x\| + \|Tx_n - Tx\| \leq \varepsilon$$

Therefore, $x_n$ converge to $x$ in $(X, \|\cdot\|_T)$, which implies $(X, \|\cdot\|_T)$ is a Banach space.

**3.** Let $X$ be normed space and $F$ be its a closed subspace. Define a natural map $T : F^{**} \to X^{**}$ by

$$T(a)(\phi) := a(\phi|_F)$$

for $a \in F^{**}$ and $\phi \in X^*$, where $\phi|_F$ denotes the restriction of $\phi$ on $F$.

(a) Show that $T$ is an isometry.

(b) State the definition of a reflexive space.

(c) Show that if $X$ is reflexive, then so is $F$.

**Proof.**

(a) Let $\phi \in X^*$ with $\|\phi\| \le 1$, since $|\phi|_F(x)| = |\phi(x)| \le \|\phi\|\|x\|, \forall x \in F$, then $\phi|_F \in F^*$ and $\|\phi|_F\| \le \|\phi\| \le 1$. Thus, T is well-defined and it is obvious that $T$ is linear. Moreover, for any $a \in F^{**}, \phi \in X^*$,

$$|T(a)(\phi)| = |a(\phi|_F)| \le \|a\|\|\phi|_F\| \le \|a\|\|\phi\|,$$

which yields that $T$ is bounded and $\|T\| \le 1$.
Now, it remains to prove $\|T\| \ge 1$. Indeed, for any $f \in F^*$, it follows from Hahn-Banach Theorem that, there exist a $\phi_f \in X^*$ such that $\phi_f|_F = f$ and $\|\phi_f\| = \|f\|$, since $F$ is a subspace of $X$. Then,

$$|a(f)| = |a(\phi_f|_F)| = |T(a)(\phi_f)| \le \sup_{\phi \in X^*, \|\phi\| \le 1} |T(a)(\phi)| = \|Ta\|.$$

which yields that $\|a\| \le \|Ta\|$.

(b) Let $X$ be a Banach space. Then $X$ is reflexive, if $X = X^{**}$ in the sense of isometry.

(c) We assume $F \subsetneqq X$ w.l.o.g., otherwise, $F = X$ is reflexive. To prove $F$ reflexive, it suffices to show that for any $a \in F^{**}$, there exists a $x \in F$ such that $a(f) = f(x), \forall f \in F^*$.
Since $X$ is reflexive, there exists a $x \in X$ such that, for any $a \in F^{**}$ and $\phi \in X^*$,

$$T(a)(\phi) := a(\phi|_F) = \phi(x).$$

We claim that $x \in F$. Indeed, if not, then $x \in X - F$ and $\delta := \inf_{y \in F} \|x - y\| > 0$, since $F$ is closed. Thus, by Hahn-Banach Theorem, there exist a $\phi \in X^*$ such that $\|\phi\| = 1, \phi(y) = 0$ ,$\forall y \in F$, and $\phi(x) = \delta$ which implies that

$$0 = a(\phi|_F)(=: T(a)(\phi)) = \phi(x) = \delta,$$

but this is a contradiction!
Therefore, $a(\phi|_F) = \phi_F(x)$. Note that the Hahn-Banach Theorem yields that for any $f \in F^*$, there exist a $\phi \in X^*$ such that $f = \phi|_F$. So, $F^{**}$ is reflexive.