inside the circle $|z| = 2$, write

$$f(z) = 3z^3 \quad \text{and} \quad g(z) = z^4 + 6.$$

Then observe that when $|z| = 2$,

$$|f(z)| = 3|z|^3 = 24 \quad \text{and} \quad |g(z)| \le |z|^4 + 6 = 22.$$

The conditions in Rouché's theorem are thus satisfied. Consequently, since $f(z)$ has three zeros, counting multiplicities, inside the circle $|z| = 2$, so does $f(z) + g(z)$. That is, equation (2) has three roots there, counting multiplicities.

EXAMPLE 2. Rouché's theorem can be used to give another proof of the fundamental theorem of algebra (Theorem 2, Sec. 58). To give the detals here, we consider a polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \ne 0) \tag{3}$$

of degree n $(n \ge 1)$ and show that it has n zeros, counting multiplicities. We write

$$f(z) = a_n z^n, \qquad g(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}$$

and let z be any point on a circle $|z| = R$, where $R > 1$. When such a point is taken, we see that

$$|f(z)| = |a_n| R^n.$$

Also,

$$|g(z)| \le |a_0| + |a_1|R + |a_2|R^2 + \cdots + |a_{n-1}|R^{n-1}.$$

Consequently, since $R > 1$,

$$|g(z)| \le |a_0|R^{n-1} + |a_1|R^{n-1} + |a_2|R^{n-1} + \cdots + |a_{n-1}|R^{n-1};$$

and it follows that

$$\frac{|g(z)|}{|f(z)|} \le \frac{|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|}{|a_n| R} < 1$$

if, in addition to being greater than unity,

$$R > \frac{|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|}{|a_n|}. \tag{4}$$

That is, $|f(z)| > |g(z)|$ when $R > 1$ and inequality (4) is satisfied. Rouché's theorem then tells us that $f(z)$ and $f(z)+g(z)$ have the same number of zeros, namely n, inside C. Hence we may conclude that $P(z)$ has precisely n zeros, counting multiplicities, in the plane.

Note how Liouville's theorem in Sec. 58 only ensured the existence of at least one zero of a polynomial; but Rouché's theorem actually ensures the existence of n zeros, counting multiplicities.

EXERCISES

1. Let C denote the unit circle $|z| = 1$, described in the positive sense. Use the theorem i Sec. 93 to determine the value of $\Delta_C \arg f(z)$ when
   (a) $f(z) = z^2$; (b) $f(z) = 1/z^2$; (c) $f(z) = (2z-1)^7/z^3$.
   Ans. (a) 4π; (b) -4π; (c) 8π.

2. Let f be a function which is analytic inside and on a positively oriented simple close contour C, and suppose that $f(z)$ is never zero on C. Let the image of C under th transformation $w = f(z)$ be the closed contour Γ shown in Fig. 114. Determine th value of $\Delta_C \arg f(z)$ from that figure; and, with the aid of the theorem in Sec. 9 determine the number of zeros, counting multiplicities, of f interior to C.
   Ans. 6π; 3.

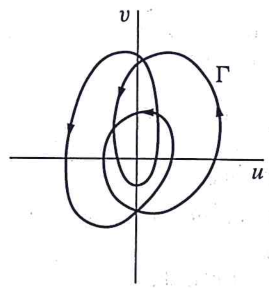

FIGURE 114

3. Using the notation in Sec. 93, suppose that Γ does not enclose the origin $w = 0$ and tha there is a ray from that point which does not intersect Γ. By observing that the absolut value of $\Delta_C \arg f(z)$ must be less than 2π when a point z makes one cycle around C and recalling that $\Delta_C \arg f(z)$ is an integral multiple of 2π, point out why the windin number of Γ with respect to the origin $w = 0$ must be zero.

4. Suppose that a function f is meromorphic in the domain D interior to a simple close contour C on which f is analytic and nonzero, and let D_0 denote the domain consistin of all points in D except for poles. Point out how it follows from the lemma in Sec. 2 and Exercise 11, Sec. 83, that if $f(z)$ is not identically equal to zero in D_0, then th zeros of f in D are all of finite order and that they are finite in number.
   Suggestion: Note that if a point z_0 in D is a zero of f that is not of finite order, the there must be a neighborhood of z_0 throughout which $f(z)$ is identically equal to zero.

5. Suppose that a function f is analytic inside and on a positively oriented simple close contour C and that it has no zeros on C. Show that if f has n zeros z_k $(k = 1, 2, \ldots, n)$ inside C, where each z_k is of multiplicity m_k, then

$$\int_C \frac{z f'(z)}{f(z)}\, dz = 2\pi i \sum_{k=1}^{n} m_k z_k.$$

   [Compare with equation (8), Sec. 93, when $P = 0$ there.]

6. Determine the number of zeros, counting multiplicities, of the polynomial
   (a) $z^6 - 5z^4 + z^3 - 2z$; (b) $2z^4 - 2z^3 + 2z^2 - 2z + 9$; (c) $z^7 - 4z^3 + z - 1$.
   inside the circle $|z| = 1$.
   Ans. (a) 4; (b) 0; (c) 3.

7. Determine the number of zeros, counting multiplicities, of the polynomial

(*a*) $z^4 - 2z^3 + 9z^2 + z - 1$; (*b*) $z^5 + 3z^3 + z^2 + 1$

inside the circle $|z| = 2$.

Ans. (*a*) 2; (*b*) 5.

8. Determine the number of roots, counting multiplicities, of the equation

$$2z^5 - 6z^2 + z + 1 = 0$$

in the annulus $1 \le |z| < 2$.

Ans. 3.

9. Show that if c is a complex number such that $|c| > e$, then the equation $cz^n = e^z$ has n roots, counting multiplicities, inside the circle $|z| = 1$.

10. Let two functions f and g be as in the statement of Rouché's theorem in Sec. 94, and let the orientation of the contour C there be positive. Then define the function

$$\Phi(t) = \frac{1}{2\pi i}\int_C \frac{f'(z) + tg'(z)}{f(z) + tg(z)}\,dz \qquad (0 \le t \le 1)$$

and follow these steps below to give another proof of Rouché's theorem.

(*a*) Point out why the denominator in the integrand of the integral defining $\Phi(t)$ is never zero on C. This ensures the existence of the integral.

(*b*) Let t and t_0 be any two points in the interval $0 \le t \le 1$ and show that

$$|\Phi(t) - \Phi(t_0)| = \frac{|t - t_0|}{2\pi}\left|\int_C \frac{fg' - f'g}{(f + tg)(f + t_0 g)}\,dz\right|.$$

Then, after pointing out why

$$\left|\frac{fg' - f'g}{(f + tg)(f + t_0 g)}\right| \le \frac{|fg' - f'g|}{(|f| - |g|)^2}$$

at points on C, show that there is a positive constant A, which is independent of t and t_0, such that

$$|\Phi(t) - \Phi(t_0)| \le A|t - t_0|.$$

Conclude from this inequality that $\Phi(t)$ is continuous on the interval $0 \le t \le 1$.

(*c*) By referring to equation (8), Sec. 93, state why the value of the function Φ is, for each value of t in the interval $0 \le t \le 1$, an integer representing the number of zeros of $f(z) + tg(z)$ inside C. Then conclude from the fact that Φ is continuous, as shown in part (*b*), that $f(z)$ and $f(z) + g(z)$ have the same number of zeros, counting multiplicities, inside C.

95. INVERSE LAPLACE TRANSFORMS

Suppose that a function F of the complex variable s is analytic throughout the finite s plane except for a finite number of isolated singularities. Then let L_R denote a vertical line segment from $s = \gamma - iR$ to $s = \gamma + iR$, where the constant γ is positive and large enough that the singularities of F all lie to the left of that segment (Fig. 115). A

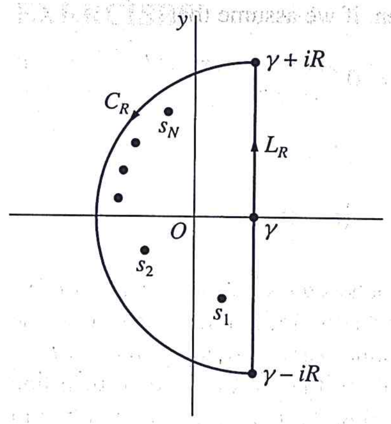

FIGURE 115

new function f of the real variable t is defined for positive values of t by means of the equation

$$f(t) = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{L_R} e^{st}F(s)\,ds \qquad (t > 0), \tag{1}$$

provided this limit exists. Expression (1) is usually written

$$f(t) = \frac{1}{2\pi i}\,\text{P.V.}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}F(s)\,ds \qquad (t > 0) \tag{2}$$

[compare with equation (3), Sec. 85], and such an integral is sometimes referred to as a ***Bromwich integral.***

It can be shown that when fairly general conditions are imposed on the functions involved, the function $f(t)$ in equation (2) is the ***inverse Laplace transform*** of the function

$$F(s) = \int_0^\infty e^{-st}f(t)\,dt, \tag{3}$$

which is the familiar Laplace transform of $f(t)$. That is, if $F(s)$ is the Laplace transform of $f(t)$, then $f(t)$ is retrieved by means of equation (2).* This is done with the aid of Cauchy's residue theorem, which tells us that

$$\int_{L_R} e^{st}F(s)\,ds = 2\pi i\sum_{n=1}^{N}\operatorname*{Res}_{s=s_n}[e^{st}F(s)] - \int_{C_R} e^{st}F(s)\,ds, \tag{4}$$

*For a detailed justification of the material in this section, see, for example, Chap. 6 of the book *Operational Mathematics*, 3rd ed., 1972, by R. V. Churchill. Also, an exceptionally clear treatment of the material appears in Chap. 7 of the book *Complex Variables with Applications*, 3rd ed., 2005, by A. D. Wunsch. Both books are listed in the Bibliography.