THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2055 Tutorial 1 (Sep 16)

Ng Wing Kit

1. Prove that $\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})=0$

Suggested Solution:

$\forall\epsilon>0$ (which reads "given any $\epsilon>0$"),
let N be the first natural number which is larger than the real number ϵ^2. Then $\forall n>N$ (which reads "for every n bigger than N"),
we have $n>\epsilon^2$,
implying $\frac{1}{\sqrt{n}}<\epsilon$.

Hence we have the following inequalities, which completes the $\epsilon-N$ proof required.

$$\begin{aligned}|\sqrt{n+1}-\sqrt{n}-0| &= (\sqrt{n+1}-\sqrt{n})(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}})\\ &= \frac{1}{\sqrt{n+1}+\sqrt{n}}\\ &< \frac{1}{\sqrt{n}}\\ &< \epsilon\end{aligned}$$

$\therefore \lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})=0$ $\square$

(Idea behind the choice of N):[1]

Actually we work 'backwards', starting from the inequality

$$|\sqrt{n+1}-\sqrt{n}-0|<\epsilon \tag{1}$$

and try to find a suitable N. (see line 2 in the 'suggested sollution'!)

One difficulty is the fact that the inequality (1) contains too many 'n's (i.e. there is the term $\sqrt{n+1}$ as well the term $\sqrt{n}$) so that we don't know how to use them to find a suitable N.

The $\boxed{\text{trick}}$ is to multiply the expression

$$\sqrt{n+1}-\sqrt{n}-0$$

by

$$\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}},$$

[1] I only write the 'idea' for this question. For questions 2 and 3, the idea is basically the same, i.e. one works backwards and try to 'clean up' the terms to make the expression contain ultimately one single n or d term.

so that the expression
$|\sqrt{n+1}-\sqrt{n}-0|$ becomes $\frac{1}{\sqrt{n+1}+\sqrt{n}}$. After doing this, we 'throw away' one term in the denominator to arrive at the inequality $\frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}}$ which motivates us how to find N.

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2. Prove that $\lim\limits_{n\to\infty}(\sqrt{n+\sqrt{n}}-\sqrt{n}) = \frac{1}{2}$.

Suggested Solution:

$\forall \epsilon > 0$,
Let N be the first natural number which is larger than the real number

$$\frac{1}{[(2\epsilon+1)^2-1]^2}.$$

Then $\forall n > N$, we have $n > \frac{1}{[(2\epsilon+1)^2-1]^2}$ hence it follows that

$$\frac{1}{2}(\sqrt{1+\frac{1}{\sqrt{n}}}-1) < \epsilon.$$

which implies the following inequalities:

$$\left|\sqrt{n+\sqrt{n}}-\sqrt{n}-\frac{1}{2}\right| = \left|\left(\sqrt{n+\sqrt{n}}-\sqrt{n}\right)\left(\frac{\sqrt{n+\sqrt{n}}+\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n}}\right)-\frac{1}{2}\right|$$

(the left-hand side of the above equality contains too many "n" terms, so we try to clean up the terms until it contains only <u>one</u> n term. To do this, we multiply by $\frac{\sqrt{n+\sqrt{n}}+\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n}}$.)

$$= \left|\frac{1}{\sqrt{1+\frac{1}{\sqrt{n}}}+1}-\frac{1}{2}\right|$$

Now we combine these two fractions to become one fraction, viz.

$$= \left|\frac{1-\sqrt{1+\frac{1}{\sqrt{n}}}}{2\sqrt{1+\frac{1}{\sqrt{n}}}+2}\right|$$

$$< \frac{1}{2}\left(\sqrt{1+\frac{1}{\sqrt{n}}}-1\right)$$

Now the above expression contains only one "n". Note that it is a positive number!

$$< \epsilon$$

$\therefore \lim_{n\to\infty}(\sqrt{n+\sqrt{n}}-\sqrt{n})=\frac{1}{2}$ $\square$

3. Let $b \in \mathbb{R}$ such that $0 < b < 1$. Prove that $\lim_{n\to\infty}(nb^n) = 0$.

Suggested Solution:

$\forall \epsilon > 0$,
as $0 < b < 1$, we can write b in the form $b = \frac{1}{1+d}$, where $d > 0$ is a positive real number.

Doing this, and requiring that n satisfies the inequality

$$\forall n > \max\left\{\frac{2}{\epsilon d^2+1},\ 3\right\}$$

we arrive at the following chain of inequalities

$$\begin{aligned}|nb^n - 0| &= \frac{n}{(1+d)^n}\\ &\text{(next, we expand this by using the Binomial Theorem.)}\\ &= \frac{n}{1+nd+\frac{n(n-1)d^2}{2!}+\sum_{i=3}^{n} C_i^n d^i}\\ &\text{(next, we throw away \underline{all} terms containing } d^3 \text{ and higher powers of } d.\text{)}\\ &< \frac{2n}{n(n-1)d^2}\\ &= \frac{2}{d^2(n-1)}\\ &< \epsilon.\end{aligned}$$

$\therefore \lim_{n\to\infty}(nb^n) = 0$ $\square$