Find and classify the critical points of the functions in Problems 1 through 22. If a computer algebra system is available, check your results by means of contour plots like those in Figs. 12.10.14–12.10.17.

1. $f(x, y) = 2x^2 + y^2 + 4x - 4y + 5$
2. $f(x, y) = 10 + 12x - 12y - 3x^2 - 2y^2$
3. $f(x, y) = 2x^2 - 3y^2 + 2x - 3y + 7$
4. $f(x, y) = xy + 3x - 2y + 4$
5. $f(x, y) = 2x^2 + 2xy + y^2 + 4x - 2y + 1$
6. $f(x, y) = x^2 + 4xy + 2y^2 + 4x - 8y + 3$
7. $f(x, y) = x^3 + y^3 + 3xy + 3$ (Fig. 12.10.14)
8. $f(x, y) = x^2 - 2xy + y^3 - y$
9. $f(x, y) = 6x - x^3 - y^3$
10. $f(x, y) = 3xy - x^3 - y^3$
11. $f(x, y) = x^4 + y^4 - 4xy$
12. $f(x, y) = x^3 + 6xy + 3y^2$
13. $f(x, y) = x^3 + 6xy + 3y^2 - 9x$ (Fig. 12.10.15)
14. $f(x, y) = x^3 + 6xy + 3y^2 + 6x$
15. $f(x, y) = 3x^2 + 6xy + 2y^3 + 12x - 24y$
16. $f(x, y) = 3x^2 + 12xy + 2y^3 - 6x + 6y$
17. $f(x, y) = 4xy - 2x^4 - y^2$ (Fig. 12.10.16)
18. $f(x, y) = 8xy - 2x^2 - y^4$
19. $f(x, y) = 2x^3 - 3x^2 + y^2 - 12x + 10$
20. $f(x, y) = 2x^3 + y^3 - 3x^2 - 12x - 3y$ (Fig. 12.10.17)
21. $f(x, y) = xy \exp(-x^2 - y^2)$
22. $f(x, y) = (x^2 + y^2) \exp(x^2 - y^2)$

In Problems 23 through 25, first show that $\Delta = f_{xx} f_{yy} - (f_{xy})^2$ is zero at the origin. Then classify this critical point by visualizing the surface $z = f(x, y)$.

23. $f(x, y) = x^4 + y^4$
24. $f(x, y) = x^3 + y^3$
25. $f(x, y) = \exp(-x^4 - y^4)$
26. Let $f(s, t)$ denote the *square* of the distance between a typical point of the line $x = t$, $y = t + 1$, $z = 2t$ and a typical point of the line $x = 2s$, $y = s - 1$, $z = s + 1$. Show that the single critical point of f is a local minimum. Hence find the closest points on these two skew lines.
27. Let $f(x, y)$ denote the square of the distance from $(0, 0, 2)$ to a typical point of the surface $z = xy$. Find and classify the critical points of f.

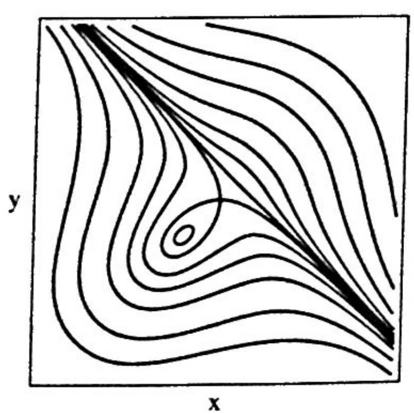

FIGURE 12.10.14 Contour plot for Problem 7.

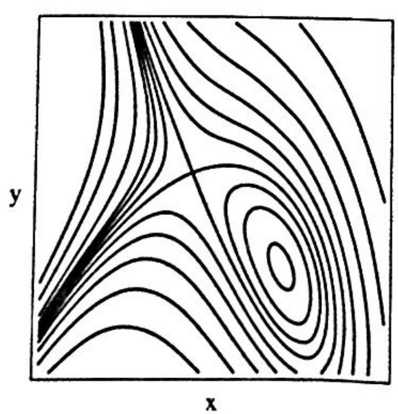

FIGURE 12.10.15 Contour plot for Problem 13.

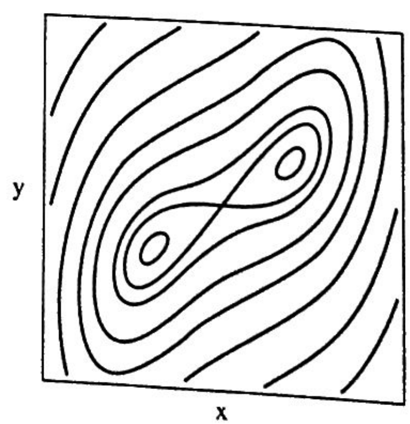

FIGURE 12.10.16 Contour plot for Problem 17.

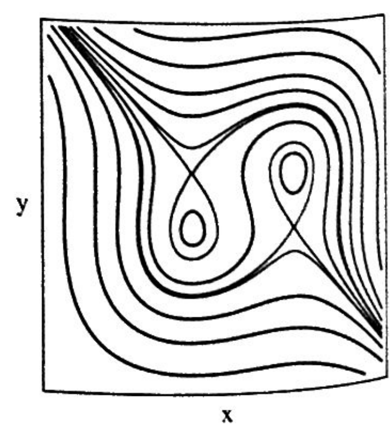

FIGURE 12.10.17 Contour plot for Problem 20.

28. Show that the surface

$$z = (x^2 + 2y^2)\exp(1 - x^2 - y^2)$$

looks like two mountain peaks joined by two ridges with a pit between them.

29. A wire 120 cm long is cut into three pieces of lengths x, y, and $120 - x - y$, and each piece is bent into the shape of a square. Let $f(x, y)$ denote the sum of the areas of these squares. Show that the single critical point of f is a local minimum. But surely it is possible to *maximize* the sum of the areas. Explain.

30. Show that the graph of the function

$$f(x, y) = xy\exp\left(\tfrac{1}{8}[x^2 + 4y^2]\right)$$

has a saddle point but no local extrema.

31. Find and classify the critical points of the function

$$f(x, y) = \sin\frac{\pi x}{2}\sin\frac{\pi y}{2}.$$

32. Let $f(x, y) = x^3 - 3xy^2$. (a) Show that its only critical point is $(0, 0)$ and that $\Delta = 0$ there. (b) By examining the behavior of $x^3 - 3xy^2$ on straight lines through the origin, show that the surface $z = x^3 - 3xy^2$ qualifies as a monkey saddle (Fig. 12.10.18).

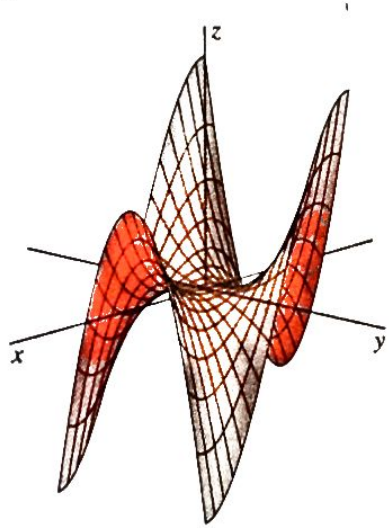

FIGURE 12.10.18 The monkey saddle of Problem 32.

33. Repeat Problem 32 with $f(x, y) = 4xy(x^2 - y^2)$. Show that near the critical point $(0, 0)$ the surface $z = f(x, y)$ qualifies as a "dog saddle" for a dog with a very short tail (Fig. 12.10.19).

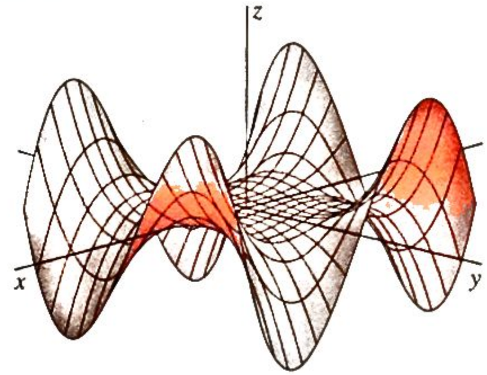

FIGURE 12.10.19 The dog saddle of Problem 33.

34. Let

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}.$$

Classify the behavior of f near the critical point $(0, 0)$.

In Problems 35 through 39, use a computer algebra program (as illustrated in the project material for this section) to approximate numerically and classify the critical point of the given function.

35. $f(x, y) = 2x^4 - 12x^2 + y^2 + 8x$

36. $f(x, y) = x^4 + 4x^2 - y^2 - 16x$

37. $f(x, y) = x^4 + 12xy + 6y^2 + 4x + 10$

38. $f(x, y) = x^4 + 8xy - 4y^2 - 16x + 10$

39. $f(x, y) = x^4 + 2y^4 - 12xy^2 - 20y^2$

Exercises 14.9

Find quadratic approximation near (0,0).

[redacted]

In Exercises 1–10, use Taylor's formula for $f(x, y)$ at the origin to find quadratic [redacted] approximations of f near the origin.

1. $f(x, y) = xe^y$
2. $f(x, y) = e^x \cos y$
3. $f(x, y) = y \sin x$
4. $f(x, y) = \sin x \cos y$
5. $f(x, y) = e^x \ln(1 + y)$
6. $f(x, y) = \ln(2x + y + 1)$
7. $f(x, y) = \sin(x^2 + y^2)$
8. $f(x, y) = \cos(x^2 + y^2)$
9. $f(x, y) = \dfrac{1}{1 - x - y}$
10. $f(x, y) = \dfrac{1}{1 - x - y + xy}$
11. Use Taylor's formula to find a quadratic approximation of $f(x, y) = \cos x \cos y$ at the origin. Estimate the error in the approximation if $|x| \le 0.1$ and $|y| \le 0.1$.
12. Use Taylor's formula to find a quadratic approximation of $e^x \sin y$ at the origin. Estimate the error in the approximation if $|x| \le 0.1$ and $|y| \le 0.1$.