Indefinite Integral :

Antiderivative : A function $F(x)$ is said to be an antiderivative of $f(x)$ if $F'(x) = f(x)$.

The process of finding antiderivatives is called indefinite integration.

e.g. If $f(x) = 2x$, $F(x) = x^2$,

then we have $F'(x) = f(x)$, so $F(x)$ is an antiderivative of $f(x)$.

However, consider $F(x) = x^2 + C$, where C is a constant.

Then, we still have $F'(x) = f(x)$.

Therefore, antiderivative of a function $f(x)$ is NOT unique.

That is why we call "an" antiderivative instead of "the" antiderivative.

Natural question : If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$, what is the relation between them ?

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what is the relation between them ?

Answer : $F(x)$ and $G(x)$ differ by a constant.

proof : Suppose $F'(x) = G'(x) = f(x)$

Let $H(x) = F(x) - G(x)$

Then $H'(x) = F'(x) - G'(x) = 0$

$\therefore$ $H(x)$ is a constant function , i.e. $H(x) = C$ for some constant C .

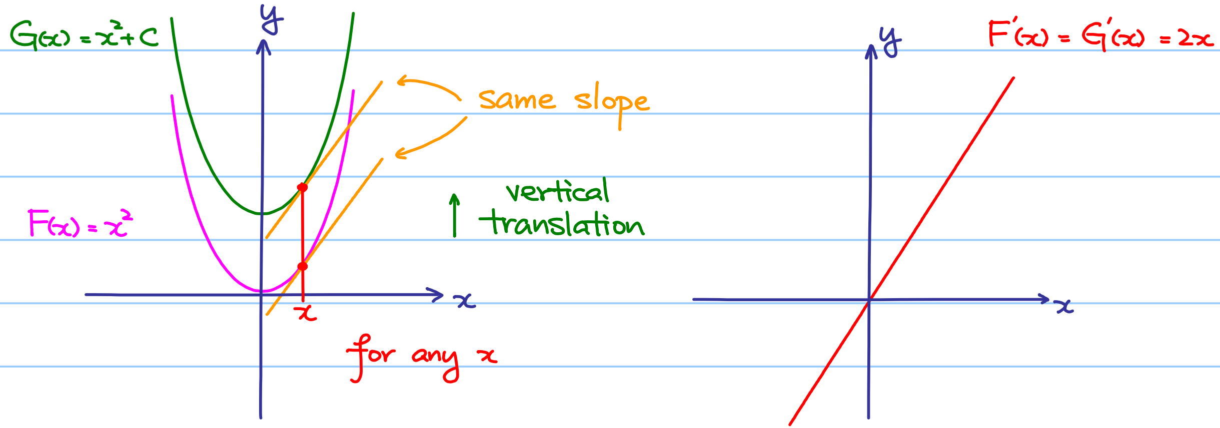
i.e. $F(x) = G(x) + C$

Therefore , antiderivative of a function $f(x)$ is NOT unique,
but it is unique up to a constant.

e.g. If $f(x) = 2x$, $F(x) = x^2$

then we have $F'(x) = f(x)$, so $F(x) = x^2$ is an antiderivative of $f(x) = 2x$

and all antiderivatives of $f(x)$ must be of the form $x^2 + C$.



If $F(x)$ is an antiderivative of $f(x)$, we write

integrand

$$\int f(x)\,dx = F(x) + C$$

integral symbol   variable of integration

e.g. $\int 2x\,dx = x^2 + C$

If $F(x)$ is an antiderivative of $f(x)$,

$$F(x) \xrightarrow{\text{differentiate}} f(x) \xrightarrow{\text{integrate}} \int f(x)\,dx = F(x) + C$$

$$= \text{original function up to a constant}$$

Note: When we write $\int f(x)\,dx$, sometimes it may be regarded as a class of functions.

Rules for Integrating Common Functions

1) $\int k\,dx = kx + C$, for constant k.

2) $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$, for all n except -1.

3) $\int \frac{1}{x}\,dx = \ln|x| + C$

4) $\int e^x\,dx = e^x + C$

5) $\int \cos x\,dx = \sin x + C$

6) $\int \sin x\,dx = -\cos x + C$

7) $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$

Algebraic Rules For Indefinite Integration

1) $\int k f(x)\,dx = k\int f(x)\,dx$

2) $\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Note : 1) $\frac{d}{dx}\left(\int k f(x)\,dx\right) = \frac{d}{dx}\left(k\int f(x)\,dx\right) = k f(x)$

i.e. $\int k f(x)\,dx$ and $k\int f(x)\,dx$ differ by a constant.

but it is absorbed by $\int$.

2) $\frac{d}{dx}\left(\int f(x) \pm g(x)\,dx\right) = \frac{d}{dx}\left(\int f(x)\,dx \pm \int g(x)\,dx\right) = f(x) \pm g(x)$.

e.g. $\int 2x^5 - 3x^2 + 7x + 5 \, dx$

$\int dx$ means $\int 1 \, dx$

$= 2\int x^5 \, dx - 3\int x^2 \, dx + 7\int x \, dx + 5\int dx$

$\int$ still there, No need to add $+C$!

$= 2\left(\frac{x^6}{6}\right) - 3\left(\frac{x^3}{3}\right) + 7\left(\frac{x^2}{2}\right) + 5x + C$

$= \frac{x^6}{3} - x^3 + \frac{7x^2}{2} - 5x + C$

e.g. $\int \frac{x^3-5}{x}\,dx$

$= \int x^2 - \frac{5}{x}\,dx$

$= \frac{x^3}{3} - 5\ln|x| + C$

e.g. Find a function $F(x)$ such that $F(0) = 3$ and $F'(x) = 2x$.

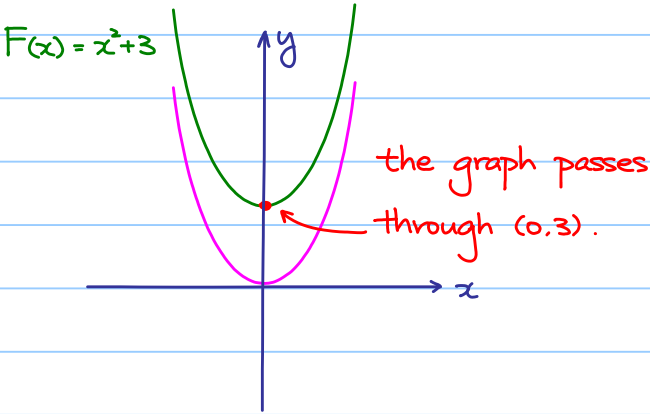
$F'(x) = 2x$

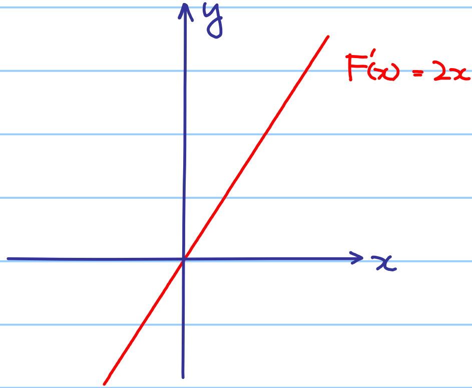
$F(x) = \int 2x \, dx$

$= x^2 + C$

$F(0) = 0^2 + C = 3 \Rightarrow C = 3$

$\therefore F(x) = x^2 + 3$


$F(x) = x^2+3$
y
x
the graph passes through (0,3).


y
$F'(x) = 2x$
x

Integration by Substitution

Question : $\int (2x+1)^{2015} dx = ?$

Hard to integrate by expanding the polynomial.

Solution : Integration by Substitution

Integration by Substitution : $\int f(u(x))\, u'(x)\, dx = \int f(u)\, du$

OR : $\int f(x) \frac{du}{dx} dx = \int f(u)\, du$

proof : $\frac{d}{dx}\int f(u(x))\,u'(x)\,dx = f(u(x))\,u'(x)$

$$\frac{d}{dx}\int f(u)\,du = \frac{d}{du}\int f(u)\,du \cdot \frac{du}{dx} \quad \text{(Chain Rule)}$$

$$= f(u(x)) \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\int f(u(x))\,u'(x)\,dx = \frac{d}{dx}\int f(u)\,du$$

$$\therefore \int f(u(x))\,u'(x)\,dx = \int f(u)\,du$$

e.g. $\int (2x+1)^{2015}\,dx = ?$

Let $u(x) = 2x+1 \qquad u'(x) = 2$

$f(u) = u^{2015} \qquad f(u(x)) = (2x+1)^{2015}$

$$\int (2x+1)^{2015}\,dx = \frac{1}{2}\int \underbrace{(2x+1)^{2015}}_{f(u(x))} \cdot \underbrace{2}_{u'(x)}\,dx = \frac{1}{2}\int \underbrace{u^{2015}}_{f(u)}\,du$$

$$= \frac{1}{4032}u^{2016} + C = \frac{1}{4032}(2x+1)^{2016} + C$$

But, usually we write,

$$\int (2x+1)^{2015}\, dx$$

$$= \int u^{2015}\, \frac{1}{2} du$$

$$= \frac{1}{4032} u^{2016} + C$$

$$= \frac{1}{4032} (2x+1)^{2016} + C$$

Let $u = 2x+1$

$$\frac{du}{dx} = 2$$

$$dx = \frac{1}{2} du$$

e.g. $\int e^{ax}\,dx$

$= \int e^{u} \cdot \frac{1}{a}\,du$

$= \frac{1}{a} e^{u} + C$

$= \frac{1}{a} e^{ax} + C$

Let $u = ax$

$\frac{du}{dx} = a$

$dx = \frac{1}{a}\,du$

e.g. $\int 6x(4x^2+3)^7\,dx$

Let $u = 4x^2+3$

$\frac{du}{dx} = 8x$

$x\,dx = \frac{1}{8}du$

$= \int 6(4x^2+3)^7\,x\,dx$

$= \int 6u^7\,\frac{1}{8}dx$

$= \frac{6}{8}\cdot\frac{1}{8}u^8 + C$

$= \frac{3}{32}(4x^2+3)^8 + C$

e.g. $\int \frac{(\ln x)^2}{x}\,dx$, $x > 0$

$\int \frac{(\ln x)^2}{x}\,dx$

$= \int u^2\,du$

$= \frac{1}{3}u^3 + C$

$= \frac{1}{3}(\ln x)^3 + C$

Let $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$\frac{1}{x}\,dx = du$

Question : How to make a guess of $u(x)$?

Integration by Substitution : $\int f(u(x))\, u'(x)\, dx = \int f(u)\, du$

e.g. $\int \frac{(\ln x)^2}{x} dx = \int (\ln x)^2 \cdot \frac{1}{x} dx$ Let $u = \ln x$

Realize the integrand as a product of parts and make a guess of $u(x)$ such that one part can be realized as a function $f(u)$, another part is $u'(x)$

Ex : 1) Show that $\int \frac{1}{ax+b} dx = \frac{1}{a} \ln|ax+b| + C$. Hint : Let $u = ax+b$

2) Evaluate

a) $\int x^3 e^{x^4} dx$ Hint : Let $u = x^4$ Ans : $\frac{1}{4} e^{x^4} + C$

b) $\int 6x\sqrt{x^2+3}\, dx$ Hint : Let $u = x^2+3$ Ans : $2(x^2+3)^{\frac{3}{2}} + C$

Integration of Exponential Functions :

Recall : $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$

In general : $\int a^x\,dx = ?$ for $a > 0$

Recall : $a^x = e^{\ln a^x} = e^{(\ln a)x}$

$$\therefore \int a^x\,dx = \int e^{(\ln a)x}\,dx$$

$$= \frac{1}{\ln a}e^{(\ln a)x} + C$$

$$= \frac{a^x}{\ln a} + C$$

OR : Recall that $\frac{d}{dx}a^x = a^x \ln a$

so $\frac{d}{dx}\frac{a^x}{\ln a} = a^x$, and $\int a^x\,dx = \frac{a^x}{\ln a} + C$

Integration of Logarithmic Functions :

$\int \ln x \, dx = ?$ for $x > 0$

Ex : $\frac{d}{dx} x\ln x - x$

Ans : $\ln x$!

Therefore , $\int \ln x \, dx = x\ln x - x + C$

Problem : How do we know $\frac{d}{dx} x\ln x - x = \ln x$ in advance ?

(Make a guess of antiderivative of $\ln x$ directly)

Any direct way to find an antiderivative of $\ln x$? (Yes, later !)

e.g. (Constant issue)

$$\int (x+1)^2\,dx \qquad \text{let } u = x+1$$

$$= \int u^2\,du \qquad du = dx$$

$$= \frac{1}{3}u^3 + C$$

$$= \frac{1}{3}(x+1)^3 + C$$

$$= \frac{1}{3}x^3 + x^2 + x + \frac{1}{3} + C$$

seems to be different !

$$\int (x+1)^2\,dx$$

$$= \int x^2 + 2x + 1\,dx$$

$$= \frac{1}{3}x^3 + x^2 + x + C$$

Ans : This C is NOT that C !

Integration of Rational Functions :

- $\int \frac{p(x)}{ax+b}\,dx$

By long division, $p(x) = (ax+b)\,q(x) + R$

$$\begin{array}{r|l} & q(x) \\ \hline ax+b & p(x) \\ & \vdots \\ \hline & R \end{array}$$

$$\frac{p(x)}{ax+b} = q(x) + \frac{R}{ax+b}$$

Then $\int \frac{p(x)}{ax+b}\,dx = \int q(x) + \frac{R}{ax+b}\,dx$

We know how to integrate !

e.g. $\int \frac{x^2+3x+5}{x+1}\,dx$

$$\begin{array}{r} x+2 \\ x+1 \,\overline{\smash{)}\, x^2+3x+5} \\ \underline{x^2+x} \\ 2x+5 \\ \underline{2x+2} \\ 3 \end{array}$$

$\therefore\ x^2+3x+5 = (x+1)(x+2)+3$

$\frac{x^2+3x+5}{x+1} = x+2+\frac{3}{x+1}$

$= \int x+2+\frac{3}{x+1}\,dx$

$= \frac{x^2}{2}+2x+3\ln|x+1|+C$

Ex : Evaluate $\int \frac{6x^2-5x+1}{3x-2}\,dx$

Ans : $x^2-\frac{x}{3}+\frac{1}{9}\ln|3x-2|+C$

- $$\int \frac{ax+b}{(r_1x+s_1)(r_2x+s_2)}\,dx$$

Express $\dfrac{ax+b}{(r_1x+s_1)(r_2x+s_2)}$ into the form $\dfrac{A}{r_1x+s_1}+\dfrac{B}{r_2x+s_2}$.

Then $$\int \frac{ax+b}{(r_1x+s_1)(r_2x+s_2)}\,dx = \int \frac{A}{r_1x+s_1}+\frac{B}{r_2x+s_2}\,dx$$

We know how to integrate!

e.g. $\int \frac{5x-7}{x^2-2x-3}\,dx$

Note : $\frac{5x-7}{x^2-2x-3} = \frac{5x-7}{(x-3)(x+1)}$

Suppose $\frac{5x-7}{(x-3)(x+1)} \equiv \frac{A}{x-3} + \frac{B}{x+1}$

$\Rightarrow \quad 5x-7 \equiv A(x+1) + B(x-3)$

$\Rightarrow \quad A = 3 \ , \ B = 2 .$

$$\int \frac{5x-7}{x^2-2x-3}\,dx = \int \frac{3}{x-3} + \frac{2}{x+1}\,dx = 3\ln|x-3| + 2\ln|x+1| + C$$

Ex : Evaluate $\int \frac{40}{x(200-x)}\,dx$

Ans : $\frac{1}{5}(\ln|x| - \ln|200-x|) + C = \frac{1}{5}\ln\left|\frac{x}{200-x}\right| + C$

- $\displaystyle\int \frac{ax+b}{(px+q)^2}\,dx$

Express $\dfrac{ax+b}{(px+q)^2}$ into the form $\dfrac{A}{(px+q)^2}+\dfrac{B}{px+q}$

Then $\displaystyle\int \frac{ax+b}{(r_1x+s_1)(r_2x+s_2)}\,dx = \int \frac{A}{(px+q)^2}+\frac{B}{px+q}\,dx$

We know how to integrate!

e.g. $\int \frac{2x-1}{(x-2)^2}\,dx$

Suppose $\frac{2x-1}{(x-2)^2} \equiv \frac{A}{(x-2)^2} + \frac{B}{x-2}$

$\Rightarrow 2x-1 \equiv A + B(x-2)$

$\Rightarrow A = 3 , B = 2$

$\int \frac{2x-1}{(x-2)^2}\,dx = \int \frac{3}{(x-2)^2} + \frac{2}{x-2}\,dx = \frac{-3}{x-2} + 2\ln|x-2| + C$

Ex : Evaluate $\int \frac{4x+2}{(2x-1)^2}\,dx$

Ans : $\frac{-2}{2x-1} + \ln|2x-1| + C$

Remarks :

If $\deg p(x) > 1$, $\int \frac{p(x)}{(r_1x+s_1)(r_2x+s_2)}\,dx = ?$

Hint : Long division .

$$\int \frac{p(x)}{(r_1x+s_1)(r_2x+s_2)}\,dx = \int q(x) + \frac{ax+b}{(r_1x+s_1)(r_2x+s_2)}\,dx$$

reduced to pervious case !

- $\int \frac{rx+s}{ax^2+bx+c}\,dx$ where $b^2-4ac<0$

$\int \frac{1}{x^2+a^2}\,dx$ let $x = au$

$= \int \frac{1}{a^2u^2+a^2}\,a\,du$ $dx = a\,du$

$= \frac{1}{a}\int \frac{1}{u^2+1}\,du$

$= \frac{1}{a}\tan^{-1}u + C$

$= \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

$$\int \frac{1}{x^2+2x+5}\,dx$$

$$= \int \frac{1}{(x+1)^2+2^2}\,dx \qquad \text{let } u = x+1$$

$$du = dx$$

$$= \int \frac{1}{u^2+2^2}\,du$$

$$= \frac{1}{2}\tan^{-1}\frac{u}{2} + C$$

$$= \frac{1}{2}\tan^{-1}\frac{x+1}{2} + C$$

$$\int \frac{4x+7}{x^2+2x+5}\,dx$$

Note: $d(x^2+2x+5) = (2x+2)dx$

and $4x+7 = 2(2x+2)+3$

$$= \int \frac{2(2x+2)+3}{x^2+2x+5}\,dx$$

$$= 2\int \frac{2x+2}{x^2+2x+5}\,dx + 3\int \frac{1}{x^2+2x+5}\,dx$$

$$= 2\ln(x^2+2x+5) + 3\left(\frac{1}{2}\tan^{-1}\frac{x+1}{2}\right) + C$$

$$= 2\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}\frac{x+1}{2} + C$$

Remarks:

If $\deg p(x) > 1$, $\int \frac{p(x)}{ax^2+bx+c}\,dx = ?$ where $b^2-4ac<0$

Hint: Long division.

Integration of Trigonometric Functions :

- $\int \tan x \, dx$ and $\int \cot x \, dx$

$$\int \tan x \, dx$$

$$= \int \frac{\sin x}{\cos x} dx \qquad \text{let } u = \cos x$$

$$= \int -\frac{1}{u} du \qquad \frac{du}{dx} = -\sin x$$

$$= -\ln|u| + C \qquad -du = \sin x \, dx$$

$$= -\ln|\cos x| + C$$

$$= \ln|\sec x| + C$$

Ex : $\int \cot x \, dx$

$$= \int \frac{\cos x}{\sin x} dx \qquad \text{let } u = \sin x$$

Ex : $\vdots$

$$= \ln|\sin x| + C$$

• $\int \sec x \, dx$ and $\int \csc x \, dx$, t-formula

t-formula :

Let $t = \tan \frac{x}{2}$

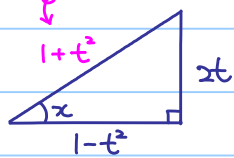
Idea : We can express all trigonometric functions in terms of t .

Note : $\tan x = \dfrac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \dfrac{2t}{1-t^2}$ and so $\cot x = \dfrac{1-t^2}{2t}$

$\therefore$ $\sin x = \dfrac{2t}{1+t^2}$ and so $\csc x = \dfrac{1+t^2}{2t}$

$\cos x = \dfrac{1-t^2}{1+t^2}$ $\sec x = \dfrac{1+t^2}{1-t^2}$

By Pyth. thm.



Therefore , all trigonometric functions in terms of t .

Note : $t = \tan \frac{x}{2}$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}(1+t^2)$$

$$dx = \frac{2}{1+t^2}\,dt$$

Idea: $\int f(\sin x, \cos x)\, dx$

$$= \int \underbrace{f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \frac{2}{1+t^2}}_{\text{Rational functions of } t.} dt$$

Transforming an integral of trigonometric function to an integral of rational function.

$$\int \csc x \, dx$$

$$= \int \frac{1+t^2}{2t} \frac{2}{1+t^2} dt$$

$$= \int \frac{1}{t} dt$$

$$= \ln|t| + C$$

$$= \ln\left|\tan\frac{x}{2}\right| + C$$

$$\int \sec x \, dx$$

$$= \int \frac{1+t^2}{1-t^2} \frac{2}{1+t^2} \, dt$$

$$= \int \frac{2}{1-t^2} \, dt$$

$$= \int \frac{1}{1+t} + \frac{1}{1-t} \, dt$$

$$= \ln|1+t| - \ln|1-t| + C$$

$$= \ln\left|\frac{1+t}{1-t}\right| + C$$

$$= \ln\left|\frac{2t+1-t^2}{1-t^2}\right| + C$$

$$= \ln|\tan x + \sec x| + C$$

e.g. $\int \frac{1}{1+\cos x}\,dx$

$$\int \frac{1}{1+\cos x}\,dx$$

$$= \int \frac{1}{1+\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2}\,dt$$

$$= \int dt$$

$$= t + C$$

$$= \tan\frac{x}{2} + C$$