$$\lim_{x\to 0} \frac{\sin x}{x} = 1 :$$

1) Consider $0 < x < \frac{\pi}{2}$, we have

Area of $\triangle OAB$ < Area of sector OAB < Area of $\triangle OAC$

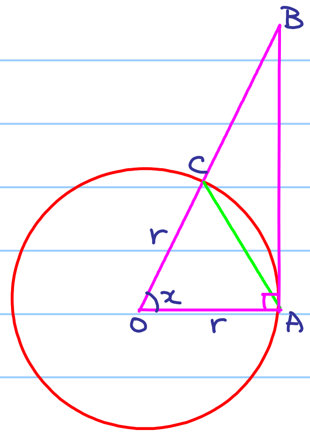
$$\frac{1}{2}r^2 \sin x < \frac{1}{2}r^2 x < \frac{1}{2}r^2 \tan x$$

$$\sin x < x < \tan x$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\frac{\sin x}{x} < 1 \qquad\qquad \cos x < \frac{\sin x}{x}$$

$$\therefore \cos x < \frac{\sin x}{x} < 1$$



2) Consider $-\frac{\pi}{2} < x < 0$, we have

Let $y = -x$, then $0 < y < \frac{\pi}{2}$, so

$$\cos y < \frac{\sin y}{y} < 1$$

$$\cos(-x) < \frac{\sin(-x)}{-x} < 1$$

$$\therefore \quad \cos x < \frac{\sin x}{x} < 1$$

Sandwich Theorem $\Rightarrow \lim_{x \to 0} \frac{\sin x}{x} = 1$

e.g. Find $\lim\limits_{x \to 0} \dfrac{\sin 3x}{2x}$.

$$\lim_{x \to 0} \frac{\sin 3x}{2x} = \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot \frac{3}{2} = 1 \cdot \frac{3}{2} = \frac{3}{2}$$

e.g. Find $\lim_{x \to 0} \frac{\cos ax - \cos bx}{x^2}$

e.g. Find $\lim_{x \to 0} \frac{\cos ax - \cos bx}{x^2}$

$$\lim_{x \to 0} \frac{\cos ax - \cos bx}{x^2} = \lim_{x \to 0} \frac{2 \sin \frac{a+b}{2} x \sin \frac{b-a}{2} x}{x^2}$$

$$= \lim_{x \to 0} 2\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right) \frac{\sin \frac{a+b}{2} x}{\frac{a+b}{2} x} \frac{\sin \frac{b-a}{2} x}{\frac{b-a}{2} x}$$

$$= \frac{b^2 - a^2}{2}$$

Limit at Infinity:

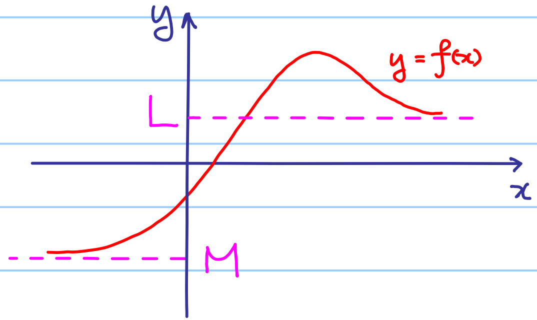
If $f(x)$ gets closer and closer to a real number L as x gets bigger and bigger (as x goes to $+\infty$), then L is called the limit of $f(x)$ at $+\infty$.

We write $\lim_{x \to +\infty} f(x) = L$.

(Similar definition for $\lim_{x \to -\infty} f(x)$)

$\lim_{x \to +\infty} f(x) = L$

$\lim_{x \to -\infty} f(x) = M$



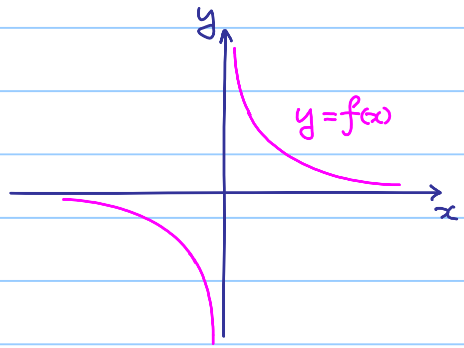
$\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ are NOT necessary to be the same!

But if $\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = L$, some simply write $\lim_{x \to \infty} f(x) = L$.

e.g. $f(x) = \frac{1}{x}$

$\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = 0$

OR simply $\lim_{x \to \infty} f(x) = 0$



FACT (without proof)

If $k > 0$, then $\lim_{x \to +\infty} \frac{1}{x^k} = 0$

Algebraic Properties of Limits at Infinity :

If $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to+\infty} g(x)$ exist (very important !) , then

(1) $$\lim_{x\to+\infty} (f(x)+g(x)) = \lim_{x\to+\infty} f(x) + \lim_{x\to+\infty} g(x)$$

(2) $$\lim_{x\to+\infty} (f(x)-g(x)) = \lim_{x\to+\infty} f(x) - \lim_{x\to+\infty} g(x)$$

(3) $$\lim_{x\to+\infty} (f(x)g(x)) = \lim_{x\to+\infty} f(x) \cdot \lim_{x\to+\infty} g(x)$$

(4) $$\lim_{x\to+\infty} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to+\infty} f(x)}{\lim_{x\to+\infty} g(x)} \quad \text{if } \lim_{x\to+\infty} g(x) \neq 0$$

Similar results hold for limits at $-\infty$.

e.g. Find $\lim_{x \to +\infty} \frac{3x^2}{x^2+x+1}$

$$\lim_{x \to +\infty} \frac{3x^2}{x^2+x+1} \neq \frac{\lim_{x \to +\infty} 3x^2}{\lim_{x \to +\infty} x^2+x+1}$$

Both limit do NOT exist!

$$= \lim_{x \to +\infty} \frac{3}{1+\frac{1}{x}+\frac{1}{x^2}}$$

$$= \frac{3}{1+0+0}$$

$$= 3$$

e.g. Find $\lim_{x\to+\infty} \frac{2x+1}{3x^2-2x+1}$

$$\lim_{x\to+\infty} \frac{2x+1}{3x^2-2x+1}$$

$$= \lim_{x\to+\infty} \frac{\frac{2}{x}+\frac{1}{x^2}}{3-\frac{2}{x}+\frac{1}{x^2}}$$

$$= \frac{0+0}{3-0+0}$$

$$= 0$$

In summary,

If $p(x)$ and $q(x)$ are polynomials

$p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$ with $a_m > 0$ (i.e. $\deg p(x) = m$)

$q(x) = b_n x^n + a_{n-1} x^{n-1} + \cdots + b_1 x + b_0$ with $b_n > 0$ (i.e. $\deg q(x) = n$)

then

$$\lim_{x \to +\infty} \frac{p(x)}{q(x)} = \begin{cases} +\infty & \text{if} \quad \deg p(x) > \deg q(x) \\ \dfrac{a_m}{b_m} & \text{if} \quad \deg p(x) = \deg q(x) \\ 0 & \text{if} \quad \deg p(x) < \deg q(x) \end{cases}$$

Similar result as the case in limits of sequences!

FACT (Without proof)

$\lim_{x \to +\infty} \left(1+\frac{1}{x}\right)^x$ exists!

We define $e = \lim_{x \to +\infty} \left(1+\frac{1}{x}\right)^x \approx 2.71828$ (i.e. call the limit e)

From $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Roughly speaking: As $x \to +\infty$, e^x grows "faster" than any x^k, where $k>0$

Limits Involving e :

e.g. Find $\lim_{x\to+\infty}\left(1+\frac{1}{2x-1}\right)^{x}$

$$\lim_{x\to+\infty}\left(1+\frac{1}{2x-1}\right)^{x} = \lim_{x\to+\infty}\left(1+\frac{1}{2x-1}\right)^{\frac{1}{2}(2x-1)+\frac{1}{2}}$$

$$= \lim_{x\to+\infty}\left[\left(1+\frac{1}{2x-1}\right)^{2x-1}\right]^{\frac{1}{2}} \cdot \left(1+\frac{1}{2x-1}\right)^{\frac{1}{2}}$$

$$= e^{\frac{1}{2}} \cdot 1$$

$$= e^{\frac{1}{2}}$$

e.g. Find $\lim_{x \to -\infty} \left(1+\frac{1}{x}\right)^x$

e.g. Find $\lim_{x \to -\infty} (1+\frac{1}{x})^x$

Let $y = -x$, as $x \to -\infty$, $y \to +\infty$

$$\lim_{x \to -\infty} (1+\frac{1}{x})^x = \lim_{y \to +\infty} (1-\frac{1}{y})^{-y}$$

$$= \lim_{y \to +\infty} (\frac{y}{y-1})^y$$

$$= \lim_{y \to +\infty} (1+\frac{1}{y-1})^{y-1} \cdot (1+\frac{1}{y-1})$$

$$= e \cdot 1$$

$$= e$$

Remark: From the above example, we know $\lim_{x \to \infty} (1+\frac{1}{x})^x$.

e.g. Find $\lim_{x \to 0} (1+x)^{\frac{1}{x}}$.

Let $y = \frac{1}{x}$, as $x \to 0$, $y \to \infty$ (Not only $+\infty$, but also $-\infty$)

$$\lim_{x \to 0} (1+x)^{\frac{1}{x}} = \lim_{y \to \infty} \left(1+\frac{1}{y}\right)^{y} = e$$

Continuity :

A function $f(x)$ is said to be continuous at $x=c$ if $\lim\limits_{x\to c} f(x) = f(c)$.

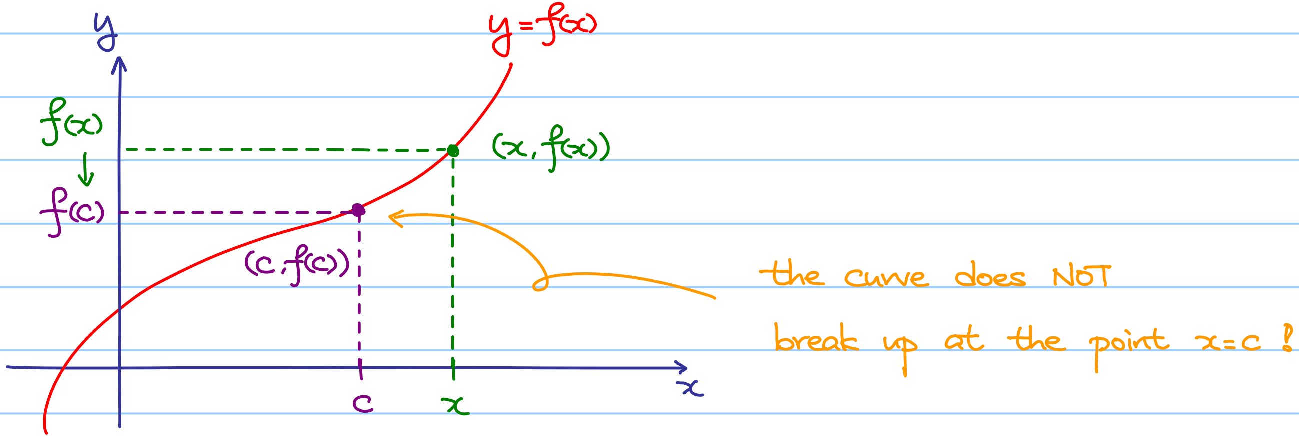
Idea :

③ they equal

$$\lim_{x\to c} f(x) = f(c)$$

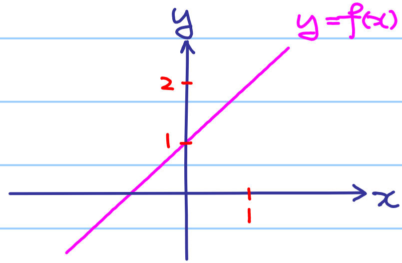
① This limit exists

② f is well-defined at $x=c$



If a function is continuous at every point,
then f is called a continuous function.

e.g. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x+1$

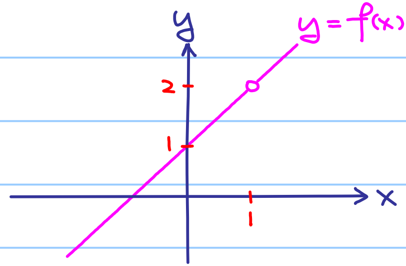


① $\lim\limits_{x \to 1} f(x) = 2$

② $f(1) = 2$

$\therefore$ f is discontinuous at $x = 1$.

e.g. Let $f : \mathbb{R} \setminus \{1\} \longrightarrow \mathbb{R}$ defined by $f(x) = \dfrac{x^2-1}{x-1}$, $x \neq 1$.



① $\lim\limits_{x \to 1} f(x) = 2$

② $f(1)$ is NOT well-defined.

$\therefore$ f is discontinuous at $x = 1$.

Recall :

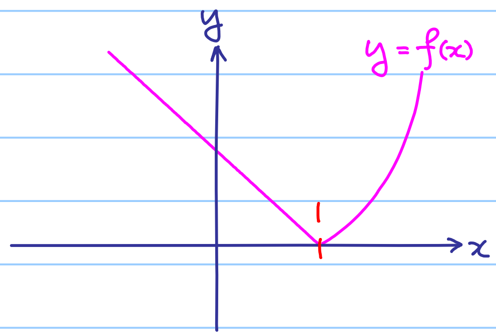
$\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$

Rewrite :

A function $f(x)$ is said to be continuous at $x = c$ if

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = f(c)$$

e.g. If $f(x) = \begin{cases} x^2-1 & \text{if } x \geq 1 \\ 1-x & \text{if } x < 1 \end{cases}$



① $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 - 1 = 0$

② $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 1 - x = 0$

③ $f(1) = 1^2 - 1 = 0$

$\therefore$ f is continuous at $x = 1$.

Absolute value :

$$|x| \overset{\text{def}}{=} \sqrt{x^2}$$

e.g. $|3| = \sqrt{3^2} = \sqrt{9} = 3$

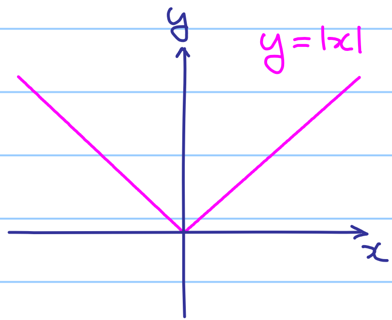
$|-3| = \sqrt{(-3)^2} = \sqrt{9} = 3$

$|0| = 0$

(Simply speaking : throw away the + or − sign)

Rewrite :

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



e.g. Prove $f(x) = |x|$ is continuous at $x = 0$.

① $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$

② $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -x = 0$

③ $f(0) = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0) = 0$

$\therefore$ $f(x)$ is continuous at $x = 0$.

Further question: Is $f(x) = |x|$ a continuous function?

Remarks :

1) We can further rewrite :

A function $f(x)$ is said to be continuous at $x=c$ if $\lim_{h\to 0} f(c+h) = f(c)$

(Hint : let $x = c+h$, as $h \to 0$, $x \to c$)

2) FACT (Without proof)

- polynomial function $p(x)$ is continuous everywhere.
- $\sqrt{x}$ is continuous for $x \geq 0$
- All trigonometric functions are continuous at every point where they are defined.
- If $f(x)$, $g(x)$ are continuous, then $f(x) \pm g(x)$, $f(x)g(x)$, $\frac{f(x)}{g(x)}$ (when $g(x) \neq 0$) are continuous.
- If $f(x)$, $g(x)$ are continuous, then $f(g(x))$ (when it is defined) is continuous.

e.g. Let $f: \mathbb{R} \to \mathbb{R}$ be a function such that

(i) f is continuous at 0.

(ii) $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

Show that :

a) $f(0) = 0$;

b) f is continuous everywhere.

a) Putting $x = y = 0$,

$$f(0+0) = f(0) + f(0)$$
$$f(0) = 2f(0)$$
$$f(0) = 0$$

b) f is continuous at 0 $\Rightarrow \lim_{h\to 0} f(0+h) = f(0)$

$\Rightarrow \lim_{h\to 0} f(h) = f(0) = 0$

Let $x_0 \in \mathbb{R}$,

$$\lim_{h\to 0} f(x_0+h) = \lim_{h\to 0} f(x_0) + f(h) \qquad \text{(Property of } f\text{)}$$
$$= f(x_0) + \lim_{h\to 0} f(h)$$
$$= f(x_0)$$

$\therefore$ f is continuous everywhere.

e.g. $f(x) = \dfrac{2x^2+3}{x^2-3x+2}$ quotient of two polynomials (continuous functions)

$= \dfrac{2x^2+3}{(x-2)(x-1)}$ the denominator is nonzero when $x \neq 1$ or 2

$\therefore$ $f(x)$ is continuous everywhere except $x = 1, 2$

Sequential Criterion for Continuity

A function f is continuous at c if and only if for every sequence a_n with $\lim_{n\to\infty} a_n = c$, we have $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(c)$.

e.g. Consider $a_n = \frac{n^2+1}{4n^2+3}$, we have $\lim_{n\to\infty} a_n = \frac{1}{4}$

Also, we know $f(x) = \sqrt{x}$ is continuous at $\frac{1}{4}$,

$\therefore \lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$

$$\lim_{n\to\infty} \sqrt{\frac{n^2+1}{4n^2+3}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

e.g. Consider

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}, \quad \text{and} \quad a_n = \frac{1}{n}.$$

Note : f is NOT continuous at $x = 0$.

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f\left(\frac{1}{n}\right) = \lim_{n\to\infty} 0 = 0$$

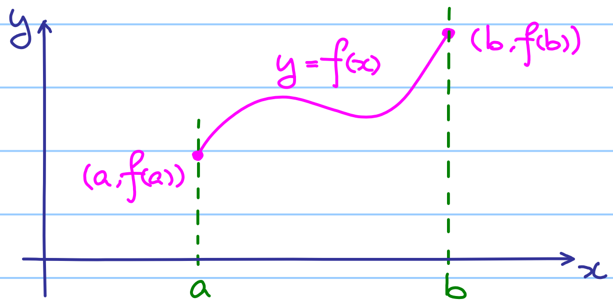
but $f\left(\lim_{n\to\infty} a_n\right) = f(0) = 1$

Continuous on [a,b] :

Let $f:[a,b]\to\mathbb{R}$ be a function ,

f is said to be continuous at $x=a$ if $\lim_{x\to a^+} f(x) = f(a)$

f is said to be continuous at $x=b$ if $\lim_{x\to b^-} f(x) = f(b)$



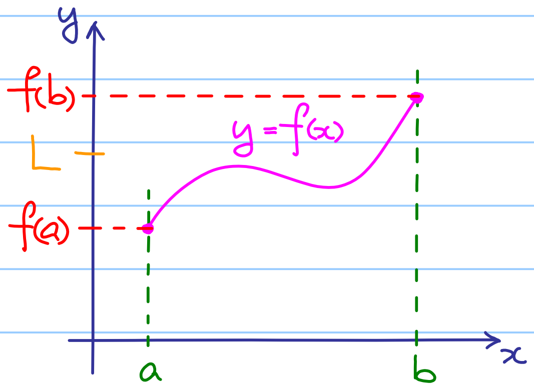
(We cannot talk about $\lim_{x\to a^-} f(x)$ and $\lim_{x\to b^+} f(x)$!)

If $f:[a,b]\to\mathbb{R}$ is continuous at every point $x\in[a,b]$,
then f is said to be continuous on $[a,b]$.

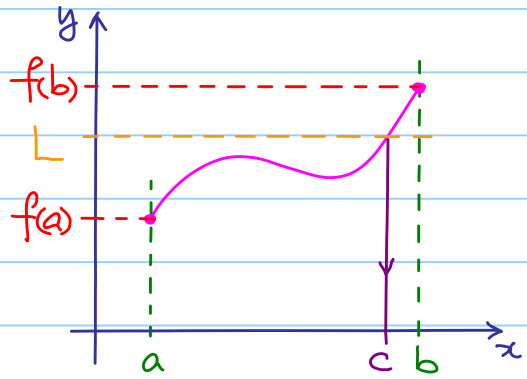
Mean Value Property (Intermediate Value Theorem):

Suppose that f is continuous on $[a,b]$ and $f(a) < f(b)$.

Furthermore, if L is a real number such that $f(a) < L < f(b)$, then there exists (at least one) $c \in (a,b)$ such that $f(c) = L$.



$\Rightarrow$



$f(c) = L$

Similar result holds for $f(a) > L > f(b)$. (What is the picture?)

e.g. x : Number of products produced (in hundreds units)

Revenue $= R(x) = 100x(400-3x^2)$

Cost $= C(x) = 120000 + 700x$

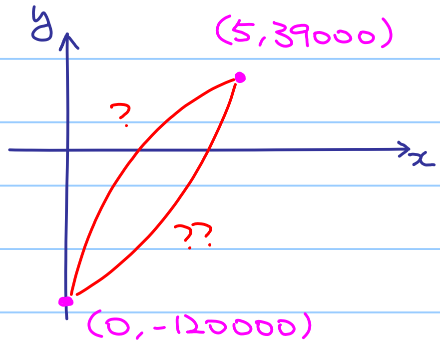
Profit $= P(x) = R(x) - C(x) = 100(-3x^3+393x-1200)$

① $P(0) = -120000 < 0$

② $P(5) = 39000 > 0$

③ $P(x)$ is a polynomial, so it is continuous everywhere,

In particular, it is continuous on $[0,5]$



We do NOT know the shape of the graph, but we know it intersects the x-axis at least once.

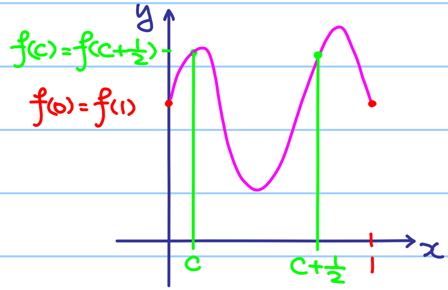
i.e. $P(c) = 0$ (which means breakeven)

for some $c \in (0,5)$

Conclusion: We do NOT know the shape of the graph, but we know it intersects the x-axis at least once, which may be enough for certain purpose.

e.g. Let $f:[0,1]\to\mathbb{R}$ be a continuous function such that $f(0)=f(1)$.

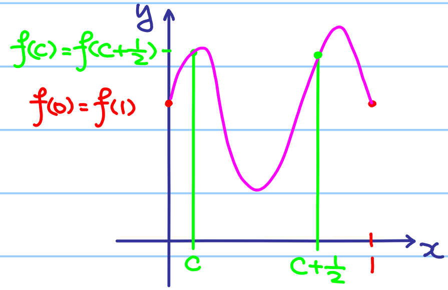
Prove that there exist $c\in[0,\frac{1}{2}]$ such that $f(c)=f(c+\frac{1}{2})$.



e.g. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$.

Prove that there exist $c \in [0, \frac{1}{2}]$ such that $f(c) = f(c+\frac{1}{2})$.

Let $g(x) = f(x) - f(x+\frac{1}{2})$ which is cont. on $[a,b]$



$g(0) = f(0) - f(\frac{1}{2})$

$g(\frac{1}{2}) = f(\frac{1}{2}) - f(1) = -g(0)$

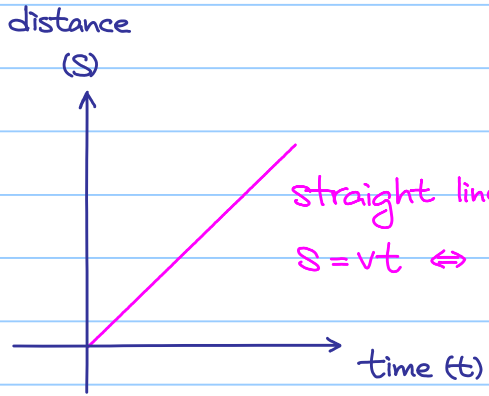
If $g(0) = 0$, done! $(c = 0)$

$$\left.\begin{array}{l} \text{If } g(0) > 0, \text{ then } g(\frac{1}{2}) < 0 \\ \quad g(0) < 0, \text{ then } g(\frac{1}{2}) > 0 \end{array}\right\} \begin{array}{l} \text{Intermediate Value Theorem} \\ \Rightarrow \exists\, c \in [0, \frac{1}{2}] \text{ s.t. } g(c) = 0 \end{array}$$

i.e. $f(c) = f(c+\frac{1}{2})$

Differentiation :

Recall : (average) speed $= \dfrac{\text{distance}}{\text{time}}$



straight line

$S = vt \Leftrightarrow v = \dfrac{S}{t}$

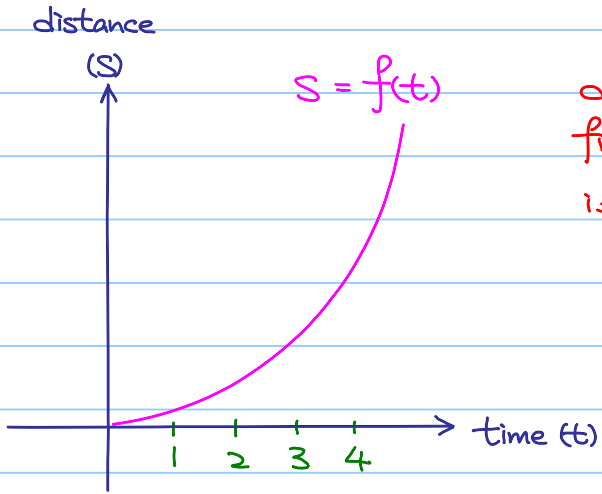
Note : constant speed !

speed = slope = v

Remark :

Using displacement and velocity if you know .

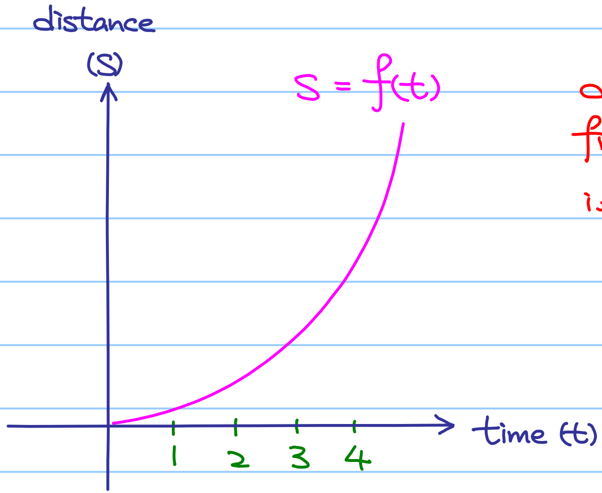
How about this case ?



distance traveled from t=0 to t=1 < distance traveled from t=3 to t=4

i.e. speed is changing

Speed is different at different moment.



distance traveled from $t=0$ to $t=1$ < distance traveled from $t=3$ to $t=4$

i.e. speed is changing

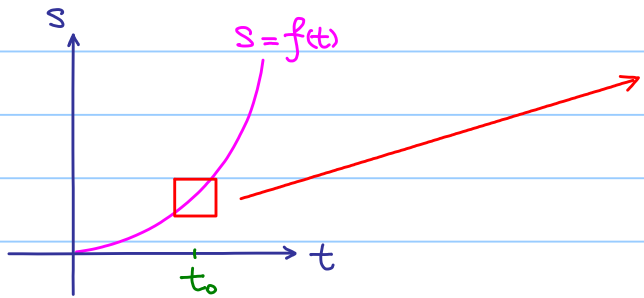
Speed is different at different moment.

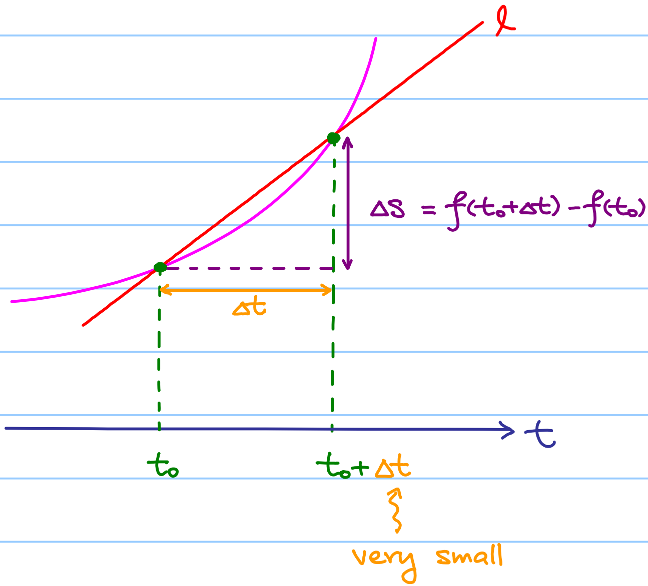
Hold on!

What is the meaning of speed at a particular moment (instantaneous speed)?

We need a definition!

Instantaneous speed at $t = t_0$:





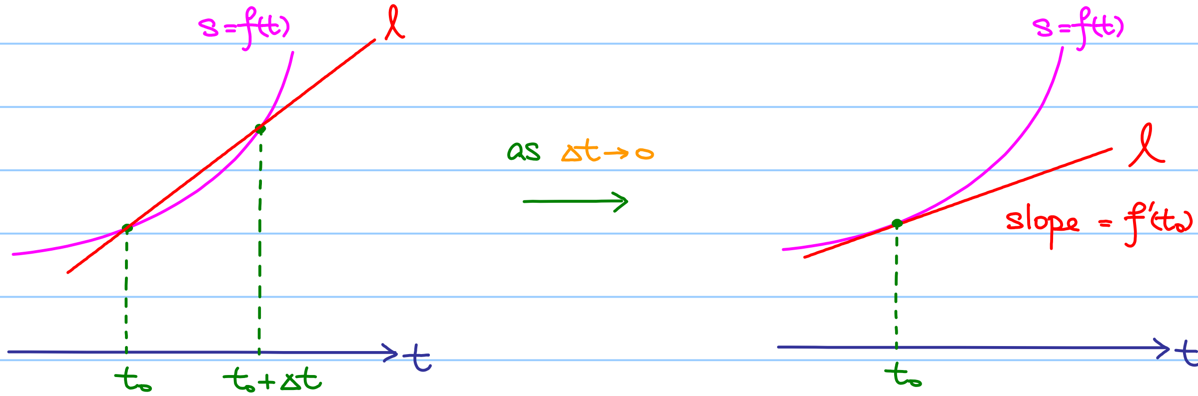
Average speed between t_0 and $t_0 + \Delta t$

$$= \frac{\text{change in distance}}{\text{change in time}} = \frac{\Delta s}{\Delta t} = \frac{f(t_0+\Delta t) - f(t_0)}{\Delta t} = \text{slope of } \ell$$

Idea : Let Δt becomes smaller and smaller !

Instantaneous speed at $t=t_0$ is defined to be $\lim\limits_{\Delta t \to 0} \dfrac{f(t_0+\Delta t)-f(t_0)}{\Delta t}$

(provided it exists , if so , it is denoted by $f'(t_0)$)



Note : When $\Delta t \to 0$, ℓ becomes the tangent line at $t=t_0$, so

slope of the tangent line at $t=t_0$ $= f'(t_0)$

e.g. If $s = f(t) = t^2$, find $f'(2)$ (instantaneous speed at $t=2$).

$$f'(2) = \lim_{\Delta t \to 0} \frac{f(2+\Delta t) - f(2)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(2+\Delta t)^2 - 2^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{4\Delta t + \Delta t^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} 4 + \Delta t = 4$$

In general, we have $y = f(x)$, fix x_0.

Then $f'(x_0)$ means rate of change of y with respect to x at $x = x_0$.

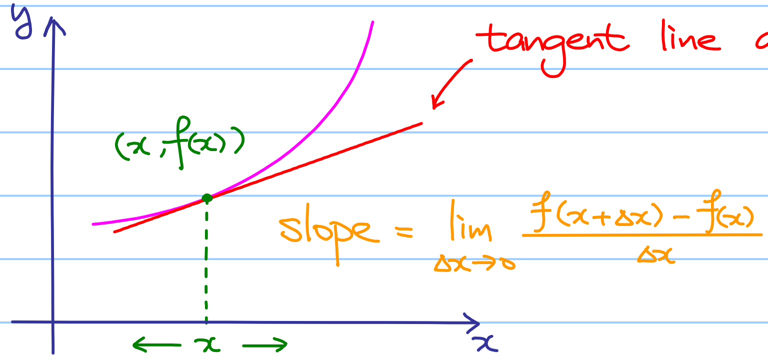
$f(x)$ is said to be differentiable at $x = x_0$ if

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \text{ exists}$$ (denoted by $f'(x_0)$)

It is called the derivative of $f(x)$ at $x = x_0$.

Note: By definition, if $f(x_0)$ is NOT well-defined, we cannot define $f'(x_0)$, so $f(x)$ must NOT be differentiable at $x = x_0$.

Perform the pervious step to different points :



Recall : What is a function ?

Roughly speaking , given an input x , return a value .

Now, we construct a new function , $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ (if exists)

(i.e. given an input x, return the slope of the tangent line at $(x, f(x))$)

e.g. If $f(x) = x^2$, find $f'(x)$

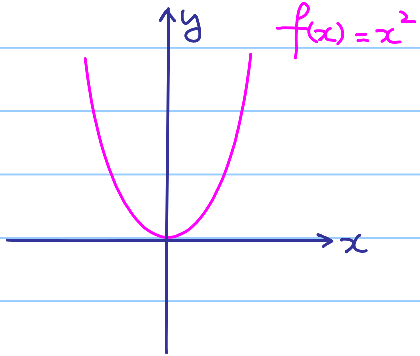
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

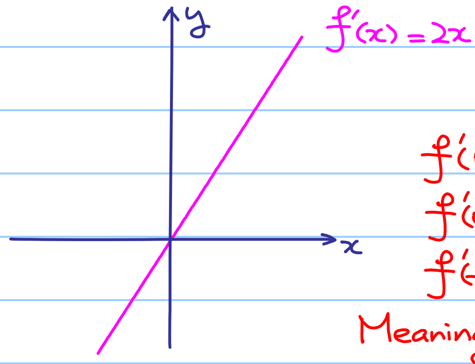
$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + \Delta x = 2x$$

Relation between the graphs of $f(x) = x^2$ and $f'(x) = 2x$:





$f'(1) = 2$

$f'(0) = 0$

$f'(-1) = -2$

Meaning ??

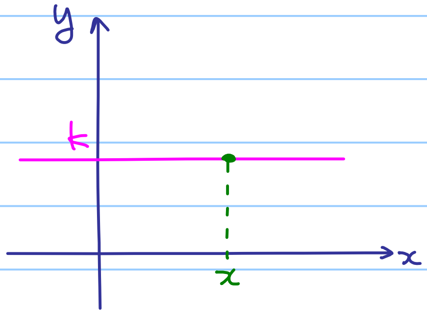
Notations :

$$y = f(x) = x^2$$

$$\frac{df}{dx} = \frac{dy}{dx} = f'(x) = 2x$$

$$\left.\frac{df}{dx}\right|_{x=3} = \left.\frac{dy}{dx}\right|_{x=3} = f'(3) = 2(3) = 6$$

e.g. If $f(x) = k$, where k is a constant , $f'(x) = ?$



Note : Slope of the tangent line at $(x, f(x)) = (x, k)$ is zero.

$\therefore f'(x) = 0$

Concrete computation :

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{k-k}{\Delta x} \quad (\Delta x \neq 0)$$

$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 0 = 0$$

Ex: Find $f'(x)$ if

(a) $f(x) = x$ Ans: $f'(x) = 1$

(b) $f(x) = x^3$ $f'(x) = 3x^2$

FACT (Without proof)

If $f(x) = x^r$, where r is a real number,

then $f'(x) = rx^{r-1}$ whenever it is defined.

(Think: If $r = \frac{1}{2}$, $f(x) = \sqrt{x}$ which is defined when $x \geq 0$)